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A METHOD FOR ESTIMATING VARIATIONS IN THE ROOTS OF THE LATERAL-STABILITY QUARTIC DUE TO CHANGES IN MASS AND AERODYNAMIC PARAMETERS OF AN AIRPLANE

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SUMMARY

A method is presented for estimating variations in the roots of the lateral-stability quartic due to changes in mass and aerodynamic parameters of an airplane. The method is applied to three high-speed airplanes and the changes of their lateral stability characteristics are determined by considering increments in various airplane parameters.

The expressions indicating the rate of change of the Dutch roll damping and frequency with respect to the stability derivatives and mass characteristics are simplified and the results when compared with those obtained from the exact expressions show very good agreement.

The rates of change of the roots with respect to the parameters are shown to have a definite relationship with the amplitude coefficients and ratios of the lateral modes of motion of the airplane subsequent to applied forces or moments. From these relationships, calculation of the rate of change of the roots with respect to five prescribed parameters allows determination of the remaining partial derivatives, the amplitude coefficients, and ratios mentioned above.

The derived expressions should afford some insight into the types of automatic stabilization devices likely to be most effective for a given airplane since, if automatic-stabilizer dynamics are neglected, the stabilizer is effectively varying one or more of the mass or aerodynamic parameters of the airplane.

A method is given in the appendix which can be used to calculate approximately the roots of the lateral-stability quartic.

INTRODUCTION

Recent investigations of the lateral stability characteristics of airplanes have indicated that small variations in the estimate of the mass and aerodynamic parameters of a given airplane may cause pronounced changes in its stability. (For example, see refs. 1 to 4.) Estimations of the mass and aerodynamic parameters of the airplane, whether from wind-tunnel data, flight tests, or existing theory, are subject to certain probable errors. Hence, a means of evaluating the effect of these anticipated probable errors on the stability characteristics of a given airplane should prove very useful. Also, such a tool should provide some insight into the relative importance of parameters or combinations of parameters affecting the stability of the airplane and, as a result, provides trends which should be useful in the selection of automatic stabilizing devices for particular airplanes.

The purpose of the present investigation is to derive expressions from which the approximate variation of the roots of the lateral-stability characteristic-quartic equation due to small changes in one or more of the airplane parameters can be calculated. As a check, the results obtained by utilizing the derived expressions are compared with the results of exact calculations of the variation of the roots due to changes in certain parameters for three different airplanes. Also, there is presented a method which can be used to calculate approximately the roots of the characteristic-quartic equation for a given set of parameters.

The rates of change of the roots of the characteristic stability quartic with respect to the airplane mass and aerodynamic parameters are shown to have a unique relationship with the amplitude coefficients and the ratios of the modes of the lateral motions of the airplane subsequent to certain disturbances.

K. Mitchell in reference 5 presented a method for calculating the approximate changes in the roots of a quartic equation due to variations in parameters upon which its coefficients depend which is very similar to the present method. The present investigation, conducted independently of that of Mitchell, makes use of a slightly different approach to the problem and, in addition, the derived expressions are examined with a view toward obtaining simpler expressions with which to work. Also, an attempt is made to establish a criterion which defines the range of variations in the parameters for which the assumption of linearity is valid.

SYMBOLS AND COEFFICIENTS

A,B,C,D,E	coefficients of lateral-stability equation
B',C',D',E'	coefficients of lateral-stability equation having been divided by A
a	measure of damping of Dutch roll oscillation
$a \pm i\omega$	complex root of stability equation corresponding to the Dutch roll oscillation
a,b,c,...i	coefficients in solutions of lateral equations of motion
b	wing span, ft
C_L	trim lift coefficient, $\frac{W \cos \gamma}{qS}$
C_l	rolling-moment coefficient, $\frac{\text{Rolling moment}}{qSb}$
C_{l_p}	damping-in-roll derivative, rate of change of rolling-moment coefficient with rolling-angular-velocity factor, $\frac{\partial C_l}{\partial \frac{pb}{2V}}$, per radian
C_{l_r}	rate of change of rolling-moment coefficient with yawing-angular-velocity factor, $\frac{\partial C_l}{\partial \frac{rb}{2V}}$, per radian
C_{l_β}	effective-dihedral derivative, rate of change of rolling-moment coefficient with angle of sideslip, $\partial C_l/\partial \beta$, per radian
C_n	yawing-moment coefficient, $\frac{\text{Yawing moment}}{qSb}$
C_{n_p}	rate of change of yawing-moment coefficient with rolling-angular-velocity factor, $\frac{\partial C_n}{\partial \frac{pb}{2V}}$, per radian

C_{n_r}	damping-in-yaw derivative, rate of change of yawing-moment coefficient with yawing-angular-velocity factor, $\frac{\partial C_n}{\partial \frac{rb}{2V}}$, per radian
C_{n_β}	directional-stability derivative, rate of change of yawing-moment coefficient with angle of sideslip, $\partial C_n/\partial \beta$, per radian
C_Y	lateral-force coefficient, $\frac{\text{Lateral force}}{qS}$
C_{Y_p}	rate of change of lateral-force coefficient with rolling-angular-velocity factor, $\frac{\partial C_Y}{\partial \frac{pb}{2V}}$, per radian
C_{Y_r}	rate of change of lateral-force coefficient with yawing-angular-velocity factor, $\frac{\partial C_Y}{\partial \frac{rb}{2V}}$, per radian
C_{Y_β}	lateral-force derivative, rate of change of lateral-force coefficient with angle of sideslip, $\partial C_Y/\partial \beta$, per radian
D_b	differential operator, d/ds_b
$F(\lambda)$	characteristic-quartic equation
g	acceleration due to gravity, ft/sec/sec
$i = \sqrt{-1}$	
K_X	nondimensional radius of gyration in roll about longitudinal stability axis, $\sqrt{{K_{X_o}}^2\cos^2\eta + {K_{Z_o}}^2\sin^2\eta}$
K_{X_o}	nondimensional radius of gyration in roll about principal longitudinal axis, k_{X_o}/b
K_{XZ}	nondimensional product-of-inertia parameter, $\left({K_{Z_o}}^2 - {K_{X_o}}^2\right)\sin\eta\cos\eta$
K_Z	nondimensional radius of gyration in yaw about vertical stability axis, $\sqrt{{K_{Z_o}}^2\cos^2\eta + {K_{X_o}}^2\sin^2\eta}$
K_{Z_o}	nondimensional radius of gyration in yaw about principal vertical axis, k_{Z_o}/b
k_{X_o}	radius of gyration in roll about principal longitudinal axis, ft

k_{Z_O}	radius of gyration in yaw about principal vertical axis, ft
m	mass of airplane, W/g, slugs
p	rolling velocity, $d\phi/dt$, radians/sec
p_1,p_2	coefficients of λ in two quadratics that constitute stability quartic
$\bar{p}$	Laplace transform variable, $f(\bar{p}) = \int_0^\infty e^{-\bar{p}t}F(t)dt$
q	dynamic pressure, $\frac{\rho V^2}{2}$, lb/sq ft
q_1,q_2	constant coefficients in two quadratics that constitute stability quartic
r	yawing velocity, $d\psi/dt$, radians/sec
S	wing area, sq ft
s_b	nondimensional time parameter based on span, Vt/b
$T_{1/2}$	time for amplitude of oscillation to change by a factor of 2 (positive value indicates a decrease to half-amplitude, negative value indicates an increase to double amplitude)
t	time, sec
V	airspeed, ft/sec
v	sideslip velocity along lateral axis, ft/sec
W	weight of airplane, lb
x_i	mass and aerodynamic parameters upon which A,B,C,D, and E depend
β	angle of sideslip, v/V, radians
γ	angle of flight path to horizontal axis, positive in a climb, deg
η	inclination of principal longitudinal axis of airplane with respect to flight path, positive when principal axis is above flight path at nose, deg

λ,λ_n nondimensional root of stability equation, $A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$

λ_1 real root of stability equation corresponding to spiral mode

λ_2 real root of stability equation corresponding to damping-in-roll mode

μ_b relative-density factor, $m/\rho Sb$

ρ mass density of air, slugs/cu ft

ϕ angle of bank, radians

ψ angle of yaw, radians

ω frequency of Dutch roll oscillation

ω_n natural frequency of Dutch roll oscillation, $\left(a^2 + \omega^2\right)^{1/2}$

ω' frequency of Dutch roll oscillation based on time in seconds, $\omega' = \frac{\omega V}{b}$, radians/sec

The subscript o is used to indicate the value of the quantity when $\Delta x_i = 0$.

ANALYSIS

Equations of Motion

The lateral nondimensional linearized airplane equations of motion for level flight ($\gamma = 0$) and controls fixed referred to the stability axes are

Rolling:

$$2\mu_b\left(K_X{}^2D_b{}^2\phi + K_{XZ}D_b{}^2\psi\right) = C_{l_\beta}\beta + \frac{1}{2}C_{l_p}D_b\phi + \frac{1}{2}C_{l_r}D_b\psi$$

Yawing:

$$2\mu_b\left(K_Z{}^2D_b{}^2\psi + K_{XZ}D_b{}^2\phi\right) = C_{n_\beta}\beta + \frac{1}{2}C_{n_p}D_b\phi + \frac{1}{2}C_{n_r}D_b\psi$$

Sideslipping:

$$2\mu_b\left(D_b\beta + D_b\psi\right) = C_{Y_\beta}\beta + \frac{1}{2} C_{Y_p}D_b\phi + \frac{1}{2} C_{Y_r}D_b\psi + C_L\phi$$

When $\phi_o e^{\lambda s_b}$ is substituted for ϕ, $\psi_o e^{\lambda s_b}$ for ψ, and $\beta_o e^{\lambda s_b}$ for β in the equations written in determinant form, λ must be a root of the stability equation,

$$F(\lambda) = A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \tag{1}$$

where

$$\left.\begin{aligned}
A &= 8\mu_b{}^3\left(K_X{}^2K_Z{}^2 - K_{XZ}{}^2\right) \\
B &= -2\mu_b{}^2\left(2K_X{}^2K_Z{}^2C_{Y_\beta} + K_X{}^2C_{n_r} + K_Z{}^2C_{l_p} - 2K_{XZ}{}^2C_{Y_\beta} - K_{XZ}C_{l_r} - K_{XZ}C_{n_p}\right) \\
C &= \mu_b\Big(K_X{}^2C_{n_r}C_{Y_\beta} + 4\mu_bK_X{}^2C_{n_\beta} + K_Z{}^2C_{l_p}C_{Y_\beta} + \frac{1}{2} C_{n_r}C_{l_p} - K_{XZ}C_{l_r}C_{Y_\beta} - \\
&\qquad 4\mu_bK_{XZ}C_{l_\beta} - C_{n_p}K_{XZ}C_{Y_\beta} + K_{XZ}C_{n_\beta}C_{Y_p} - K_X{}^2C_{Y_r}C_{n_\beta} + K_{XZ}C_{Y_r}C_{l_\beta} - \\
&\qquad \frac{1}{2} C_{n_p}C_{l_r} - K_Z{}^2C_{Y_p}C_{l_\beta}\Big) \\
D &= -\frac{1}{4} C_{n_r}C_{l_p}C_{Y_\beta} - \mu_bC_{l_p}C_{n_\beta} + \frac{1}{4} C_{n_p}C_{l_r}C_{Y_\beta} + \mu_bC_{n_p}C_{l_\beta} + \\
&\qquad 2\mu_bC_LK_{XZ}C_{n_\beta} - 2\mu_bC_LK_Z{}^2C_{l_\beta} + \frac{1}{4} C_{l_p}C_{n_\beta}C_{Y_r} - \frac{1}{4} C_{n_p}C_{l_\beta}C_{Y_r} - \\
&\qquad \frac{1}{4} C_{l_r}C_{n_\beta}C_{Y_p} + \frac{1}{4} C_{n_r}C_{l_\beta}C_{Y_p} \\
E &= \frac{1}{2} C_L\left(C_{n_r}C_{l_\beta} - C_{l_r}C_{n_\beta}\right)
\end{aligned}\right\} \tag{2}$$

Derivation of Expressions for the Rate of Change of the Roots With Changes in Airplane Parameters

The lateral-stability quartic is given by equation (1). The coefficients A, B, C, D, and E have been shown to be functions of the various mass and aerodynamic parameters of an airplane, henceforth designated as $x_1, x_2, x_3, \ldots x_i$. If the parameters x_i are assumed to be independent variables, a change in a parameter will result in changes in the coefficients of equation (1), that is,

$$\left.\begin{aligned}
A &= A_o + \frac{\partial A}{\partial x_i}\Delta x_i \\
B &= B_o + \frac{\partial B}{\partial x_i}\Delta x_i \\
C &= C_o + \frac{\partial C}{\partial x_i}\Delta x_i \\
D &= D_o + \frac{\partial D}{\partial x_i}\Delta x_i \\
E &= E_o + \frac{\partial E}{\partial x_i}\Delta x_i
\end{aligned}\right\} \qquad (3)$$

The roots of equation (1) will then become $\lambda = \lambda_o + \Delta\lambda$. For $\Delta x_i = 0$, the increment in the root $\Delta\lambda$ is equal to zero and $A = A_o$, $B = B_o$, $C = C_o$, $D = D_o$, and $E = E_o$. Substitution of equations (3) for A, B, C, D, and E and $\lambda = \lambda_o + \Delta\lambda$ into equation (1) and consideration of the limiting case for which Δx_i and $\Delta\lambda$ approach zero can be shown to be equivalent to differentiating equation (1) implicitly with respect to x_i. Therefore,

$$\frac{\partial \lambda}{\partial x_i} = -\frac{\frac{\partial A}{\partial x_i}\lambda_o^4 + \frac{\partial B}{\partial x_i}\lambda_o^3 + \frac{\partial C}{\partial x_i}\lambda_o^2 + \frac{\partial D}{\partial x_i}\lambda_o + \frac{\partial E}{\partial x_i}}{4A_o\lambda_o^3 + 3B_o\lambda_o^2 + 2C_o\lambda_o + D_o} \qquad (4)$$

A point of interest is that equations (3) represent the exact changes in the coefficients regardless of the size of Δx_i if the coefficients are linear functions of the parameter x_i, which is true for all the parameters except K_{XZ}.

Equation (4) expresses the slope or rate of change of the roots of equation (1) with respect to any parameter x_i. If several parameters are varied simultaneously, the total change in λ is given by the expression

$$d\lambda = \frac{\partial\lambda}{\partial x_1} dx_1 + \frac{\partial\lambda}{\partial x_2} dx_2 + \ldots + \frac{\partial\lambda}{\partial x_i} dx_i \tag{5}$$

or

$$\Delta\lambda \approx \frac{\partial\lambda}{\partial x_1} \Delta x_1 + \frac{\partial\lambda}{\partial x_2} \Delta x_2 + \ldots + \frac{\partial\lambda}{\partial x_i} \Delta x_i$$

General Form of Variation Expressions and Criterion for Determining Approximate Range of Linearity

If the coefficients of the stability quartic, the roots of this quartic, and the partials of the coefficients with respect to a selected parameter are known, the rate of change of any root with respect to this parameter can be found from equation (4). The values of the coefficients for a particular airplane can be found by evaluating equations (2). The roots may be found by either solving the characteristic quartic by various conventional methods (for example, ref. 6) or by using the approximation method presented in the appendix. The general forms of the partial of the coefficients with respect to several of the parameters x_i are given in table I. The parameters that are used are C_{l_p}, C_{l_r}, C_{l_β}, C_{n_p}, C_{n_r}, C_{n_β}, C_{Y_β}, η, ${K_{X_o}}^2$, and ${K_{Z_o}}^2$. The parameters ${K_{X_o}}^2$ and ${K_{Z_o}}^2$ (the principal radii of gyration squared) were chosen rather than ${K_X}^2$ and ${K_Z}^2$ since errors in ${K_X}^2$ and ${K_Z}^2$ could result from errors in either ${K_{X_o}}^2$, ${K_{Z_o}}^2$, or η.

After the parameter is chosen and the values of the coefficients and the values of the partials of these coefficients are substituted, equation (4) becomes a function of λ. Now depending on the nature of the λ of interest, equation (4) may be real or complex.

For λ real, the approximate change of λ due to a change in one of the above parameters is

$$\Delta\lambda \approx \frac{\partial\lambda}{\partial x_i}\Delta x_i \tag{6}$$

and the new root due to the Δx_i is approximately

$$\lambda \approx \lambda_o + \frac{\partial\lambda}{\partial x_i}\Delta x_i \tag{7}$$

For λ complex, that is, $a \pm i\omega$, equation (4) becomes complex, and by equating real to real and imaginary to imaginary, the condition exists that

$$\frac{\partial\lambda}{\partial x_i} = \frac{\partial a}{\partial x_i} + i\frac{\partial\omega}{\partial x_i} \tag{8}$$

and the approximate change of a and ω due to a change in one of the parameters is

$$\Delta a \approx \frac{\partial a}{\partial x_i}\Delta x_i \tag{9a}$$

$$\Delta\omega \approx \frac{\partial\omega}{\partial x_i}\Delta x_i \tag{9b}$$

Then, the new complex root due to the Δx_i is approximately

$$\lambda \approx a_o + \frac{\partial a}{\partial x_i}\Delta x_i + i\left(\omega_o + \frac{\partial\omega}{\partial x_i}\Delta x_i\right) \tag{10}$$

It may be well to mention here that equation (4) evaluated for $\lambda = a - i\omega$ will merely give the conjugate result of equation (4) evaluated for $\lambda = a + i\omega$, and the change in $-\omega$ may be expressed as

$$-\Delta\omega \approx -\frac{\partial\omega}{\partial x_i}\Delta x_i$$

or

$$\Delta\omega \approx \frac{\partial\omega}{\partial x_i} \Delta x_i$$

which is identical to equation (9b). Therefore, in the remaining discussion when the complex root or its slope is discussed, the complex root is assumed to be $a + i\omega$.

Naturally, the slope expression will not accurately approximate the change of λ for large values of Δx_i. Therefore, a criterion must be established that will indicate the value of Δx_i at which the expression $\lambda_o + \frac{\partial\lambda}{\partial x_i} \Delta x_i$ no longer gives a suitable approximation to the new root.

Since λ is a function of x_i, $\lambda(x_i + \Delta x_i)$ may be obtained from the Taylor's series expansion in the vicinity of x_i. The first three terms of the expansion are

$$\lambda(x_i + \Delta x_i) = \lambda(x_i) + \lambda'(x_i)\Delta x_i + \frac{\lambda''(x_i)(\Delta x_i)^2}{2!} + \ldots \tag{11}$$

where λ', λ'', and so forth, denote differentiation of λ with respect to x_i. In the previous part of this paper the third term and all succeeding terms in the above expansion were assumed negligible. Now it is assumed that, whenever the third term becomes sufficiently large to be 10 percent of the sum of the first two terms, the initial assumption of linearity no longer presents a suitable approximation to the new root. This criterion may be expressed as

$$\left| \frac{\frac{1}{2} \frac{\partial^2\lambda}{\partial x_i^2}(\Delta x_i)^2}{\lambda_o + \frac{\partial\lambda}{\partial x_i} \Delta x_i} \right| < 0.10 \tag{12}$$

The second partial of λ with respect to x_i may be obtained by differentiation of equation (4).

$$\frac{\partial^2\lambda}{\partial x_i^{\,2}} = \frac{-2\,\frac{\partial\lambda}{\partial x_i}\left[(6A\lambda^2 + 3B\lambda + C)\frac{\partial\lambda}{\partial x_i} + 4a_1\lambda^3 + 3a_2\lambda^2 + 2a_3\lambda + a_4\right]}{4A\lambda^3 + 3B\lambda^2 + 2C\lambda + D}$$

When $\lambda = a + i\omega$, the above expression becomes a complex number and $\frac{\partial^2 a}{\partial x_i^{\,2}}$ and $\frac{\partial^2\omega}{\partial x_i^{\,2}}$ are equal to the real and imaginary parts of this number, respectively. In order to apply the criterion (eq. (12)) for $\lambda = a + i\omega$, the terms a and ω must be considered separately; that is, for a change in ω equation (12) becomes

$$\left|\frac{\frac{1}{2}\,\frac{\partial^2\omega}{\partial x_i^{\,2}}(\Delta x_i)^2}{\omega_o + \frac{\partial\omega}{\partial x_i}\,\Delta x_i}\right| < 0.10$$

Since equation (12) employs only three terms of equation (11) it is possible that the Δx_i calculated from this expression will actually give slightly more or less than 10 percent. Although no results are presented, this criterion was found to be adequate for the prediction of nonlinearity of the curves.

Simplification of the Slope Equation for the Oscillatory Root

Although variations in the mass and aerodynamic parameters of an airplane may cause changes in all the roots of the stability quartic, usually changes in the oscillatory root are of primary interest; that is, the change in the damping and frequency of the Dutch roll oscillation of the airplane is of most concern. Therefore, an attempt is made to simplify equation (4), for the case where λ_o is equal to the oscillatory root $(a \pm i\omega)$, in order to reduce the amount of work required to obtain values for $\partial a/\partial x_i$ and $\partial\omega/\partial x_i$ and also to provide some insight into the parameters which affect these partial derivatives.

The denominator of equation (4) is merely the derivative of equation (1) with respect to λ. If the roots of equation (1) are, as in the usual case, two real roots, namely, λ_1 and λ_2, and a pair of conjugate complex roots, $a \pm i\omega$, the denominator of equation (4) may be written as

$$F'(\lambda) = A_o\left[(\lambda^2 - 2a\lambda + a^2 + \omega^2)(\lambda - \lambda_1) + (\lambda^2 - 2a\lambda + a^2 + \omega^2)(\lambda - \lambda_2) + (\lambda - \lambda_1)(\lambda - \lambda_2)(2\lambda - 2a)\right] \tag{13}$$

Now, if $\lambda = a + i\omega$ is substituted into equation (13),

$$F'(a + i\omega) = A_o\left[(a + i\omega - \lambda_1)(a + i\omega - \lambda_2)(2\omega i)\right] \tag{14}$$

as was pointed out in reference 5.

Also, if $\lambda = a + i\omega$, the general form of equation (4) is

$$\frac{\partial(a + i\omega)}{\partial x_i} = \frac{d + ci}{e + fi}$$

By rationalizing the above equation and by using equation (8),

$$\frac{\partial a}{\partial x_i} = \frac{de + cf}{e^2 + f^2} \tag{15}$$

$$\frac{\partial \omega}{\partial x_i} = \frac{ce - df}{e^2 + f^2} \tag{16}$$

The exact form of d and c from equation (4) are

$$d = -\frac{\partial A}{\partial x_i}(a^4 - 6a^2\omega^2 + \omega^4) - \frac{\partial B}{\partial x_i}(a^3 - 3a\omega^2) - \frac{\partial C}{\partial x_i}(a^2 - \omega^2) - \frac{\partial D}{\partial x_i}(a) - \frac{\partial E}{\partial x_i}$$

$$c = -\frac{\partial A}{\partial x_i}(4a^3\omega - 4a\omega^3) - \frac{\partial B}{\partial x_i}(3a^2\omega - \omega^3) - \frac{\partial C}{\partial x_i}(2a\omega) - \frac{\partial D}{\partial x_i}(\omega)$$

The form of e and f from equation (14) are

$$e = 2A_o\omega\left[\omega(\lambda_1 + \lambda_2) - 2a\omega\right]$$

$$f = 2A_o\omega\left[-a(\lambda_1 + \lambda_2) + a^2 - \omega^2 + \lambda_1\lambda_2\right]$$

The simplifications are now made that a and λ_1 are approximately zero compared with λ_2 and ω; that is, the damping of the Dutch roll oscillation and spiral mode are negligible compared with the damping in roll and frequency. Hence, d, c, e, and f become

$$d = -\frac{\partial A}{\partial x_i}\omega^4 + \frac{\partial C}{\partial x_i}\omega^2 - \frac{\partial E}{\partial x_i}$$

$$c = \frac{\partial B}{\partial x_i}\omega^3 - \frac{\partial D}{\partial x_i}\omega$$

$$e = 2A_o\omega^2\lambda_2$$

$$f = -2A_o\omega^3$$

Then, equations (15) and (16) may be written as

$$\frac{\partial a}{\partial x_i} = \frac{\lambda_2\left(-\frac{\partial A}{\partial x_i}\omega^4 + \frac{\partial C}{\partial x_i}\omega^2 - \frac{\partial E}{\partial x_i}\right) - \omega^2\left(\frac{\partial B}{\partial x_i}\omega^2 - \frac{\partial D}{\partial x_i}\right)}{2A_o\omega^2\left(\lambda_2{}^2 + \omega^2\right)} \tag{17}$$

$$\frac{\partial\omega}{\partial x_i} = \frac{\lambda_2\left(\frac{\partial B}{\partial x_i}\,\omega^2 - \frac{\partial D}{\partial x_i}\right) + \left(-\,\frac{\partial A}{\partial x_i}\,\omega^4 + \frac{\partial C}{\partial x_i}\,\omega^2 - \frac{\partial E}{\partial x_i}\right)}{2A_o\omega(\lambda_2^2 + \omega^2)} \tag{18}$$

Unfortunately, because of differences in sign of first-order terms which make second-order terms important, expressions (17) and (18) will not hold for a few of the parameters. Separate expressions are presented for these parameters in which some second-order terms (involving a and powers of a) are retained.

It was found that, for the three airplanes used in this paper, equation (17) may be used to calculate the rate of change of a with respect to C_{n_r}, C_{l_r}, C_{n_β}, C_{l_β}, C_{n_p}, C_{Y_β}, and η with good results.

For C_{l_p} use

$$\frac{\partial a}{\partial C_{l_p}} = \frac{\lambda_2\left[\frac{\partial C}{\partial C_{l_p}}\,\omega^2 + a\left(\frac{\partial B}{\partial C_{l_p}}\,3\omega^2 - \frac{\partial D}{\partial C_{l_p}}\right)\right] - \omega^2\left[\frac{\partial B}{\partial C_{l_p}}(\omega^2 - 3a^2) - \frac{\partial D}{\partial C_{l_p}} - \frac{\partial C}{\partial C_{l_p}}(2a)\right]}{2A_o\omega^2(\lambda_2^2 + \omega^2)} \tag{19}$$

For $K_{X_o}{}^2$ and $K_{Z_o}{}^2$ use

$$\frac{\partial a}{\partial x_i} = \frac{\lambda_2\left[\frac{\partial C}{\partial x_i} + \frac{\partial A}{\partial x_i}(6a^2 - \omega^2) + \frac{\partial B}{\partial x_i}(3a)\right] - 2a\left[\frac{\partial A}{\partial x_i}(6a^2 + \omega^2) + 3a\,\frac{\partial B}{\partial x_i}\right] + \frac{\partial D}{\partial x_i} - \frac{\partial B}{\partial x_i}\,\omega^2}{2A_o(\lambda_2^2 + \omega^2)} \tag{20}$$

Equation (18) may be used to calculate the rate of change of ω with respect to C_{n_β}, C_{n_p}, C_{l_r}, C_{l_β}, $K_{Z_o}{}^2$, and η.

For C_{n_r} and C_{Y_β} use

$$\frac{\partial\omega}{\partial x_i} = \frac{\lambda_2\left[\frac{\partial B}{\partial x_i}(\omega^2 - 3a^2) - \frac{\partial C}{\partial x_i}(2a) - \frac{\partial D}{\partial x_i}\right] + \left(\frac{\partial C}{\partial x_i}\,\omega^2 - \frac{\partial E}{\partial x_i}\right)}{2A_o\omega(\lambda_2^2 + \omega^2)} \tag{21}$$

For C_{l_p} and $K_{X_o}{}^2$ use

$$\frac{\partial\omega}{\partial x_1} = \frac{\lambda_2\left[\frac{\partial A}{\partial x_1}(4a\omega^2) + \frac{\partial B}{\partial x_1}(\omega^2 - 3a^2) - \frac{\partial C}{\partial x_1}(2a) - \frac{\partial D}{\partial x_1}\right] + \left[\frac{\partial A}{\partial x_1}(6a^2\omega^2 - \omega^4) + \frac{\partial B}{\partial x_1}(3a\omega^2) + \frac{\partial C}{\partial x_1}(\omega^2) - \frac{\partial D}{\partial x_1}(a)\right]}{2A_o\omega(\lambda_2{}^2 + \omega^2)} \tag{22}$$

The preceding expressions were simplified under the assumptions that λ_1 and a are small and hence are not expected to give a good approximation to the exact values for airplanes where the spiral root λ_1 is of the same order as the damping-in-roll root λ_2 or for airplanes which possess a large degree of damping in the Dutch roll oscillation (large $-a/\omega_n$). For an airplane where these conditions exist the general slope equation (eq. (4)) should be used.

Other Possible Simplifications for C_{n_r} and C_{n_p}

Many times, by closer examination of the separate terms of equation (17) set up for a specific parameter, an extremely simple equation may be written down which very well approximates the results obtained by the complete slope equation (eq. (4)). Two such examples using C_{n_r} and C_{n_p} are presented.

If the partial derivatives of the coefficients with respect to C_{n_r} (table I) are substituted into equation (17), the following equation is obtained:

$$\frac{\partial a}{\partial C_{n_r}} = \frac{\lambda_2\left[\mu_b\left(K_X{}^2C_{Y_\beta} + \frac{1}{2}C_{l_p}\right)\omega^2 - \frac{1}{2}C_LC_{l_\beta}\right] - \omega^2\left[\left(-2\mu_b{}^2K_X{}^2\right)\omega^2 + \frac{1}{4}\left(C_{l_p}C_{Y_\beta} - C_{l_\beta}C_{Y_p}\right)\right]}{2A_o\omega^2(\lambda_2{}^2 + \omega^2)} \tag{23}$$

Two assumptions are made: First, the term $K_X{}^2C_{Y_\beta}$ is negligible compared with $\frac{1}{2}C_{l_p}$ and, second, the term $C_{l_\beta}C_{Y_p}$ is negligible compared with $C_{l_p}C_{Y_\beta}$.

If the following two conditions are satisfied:

$$\left|C_{n_\beta}\right| > \left|\left(\frac{20C_L K_Z^2}{C_{l_p}} + \frac{K_{XZ}}{K_X^2}\right)C_{l_\beta}\right|$$

and

$$\left|C_{n_\beta}\right| > \left|\frac{2.5K_Z^2}{\mu_b K_X^2} C_{l_p} C_{Y_\beta} + \frac{K_{XZ}}{K_X^2} C_{l_\beta}\right|$$

and if A_o and λ_2 can be well-approximated by

$$A_o \approx 8\mu_b^3 K_X^2 K_Z^2$$

$$\lambda_2 \approx \frac{C_{l_p}}{4\mu_b K_X^2}$$

equation (23) reduces to

$$\frac{\partial a}{\partial C_{n_r}} \approx \frac{1}{8\mu_b K_Z^2} \tag{24}$$

This result is the same as would be obtained by assuming one degree of freedom in yaw which may be expressed as

$$\left(2\mu_b K_Z^2 D^2 - \frac{1}{2} C_{n_r} D + C_{n_\beta}\right)\psi = 0$$

where the real part of the complex root is

$$a = \frac{C_{n_r}}{8\mu_b K_Z^2}$$

and if this equation is differentiated with respect to C_{n_r}

$$\frac{\partial a}{\partial C_{n_r}} = \frac{1}{8\mu_b K_Z{}^2}$$

Equation (17) set up for C_{n_p} by using the partials of the coefficients from table I gives

$$\frac{\partial a}{\partial C_{n_p}} = \frac{\lambda_2\left[-\mu_b\left(K_{XZ}C_{Y_\beta} + \frac{1}{2}C_{l_r}\right)\omega^2\right] - \omega^2\left(2\mu_b{}^2K_{XZ}\omega^2 - \frac{1}{4}C_{l_r}C_{Y_\beta} - \mu_b C_{l_\beta} + \frac{1}{4}C_{l_\beta}C_{Y_r}\right)}{2A_o\omega^2\left(\lambda_2{}^2 + \omega^2\right)} \tag{25}$$

Again, an assumption is made as to the order of magnitude of several terms. The terms $\frac{1}{4}C_{l_\beta}C_{Y_r}$ and $-\frac{1}{4}C_{l_r}C_{Y_\beta}$ are assumed to be negligible compared with $\mu_b C_{l_\beta}$.

If the following condition is satisfied:

$$\left|C_{l_\beta}\right| > \left|\frac{2.5C_{l_p}K_Z{}^2\left(K_{XZ}C_{Y_\beta} + \frac{1}{2}C_{l_r}\right) + 10\mu_b K_X{}^2K_{XZ}C_{n_\beta}}{\mu_b\left(K_X{}^2K_Z{}^2 + K_{XZ}{}^2\right)}\right|$$

expression (25) may be well-approximated by

$$\frac{\partial a}{\partial C_{n_p}} \approx \frac{\mu_b C_{l_\beta}}{2A_o\left(\lambda_2{}^2 + \omega^2\right)} \tag{26}$$

As has been pointed out before, these two expressions were derived under the assumption that certain conditions must be satisfied and, if these conditions are not satisfied for a particular configuration, equations (23) and (25) should be used.

It is interesting to note that, from equation (24), the effect of the derivative C_{n_r} is highly dependent upon the moment of inertia in

yaw. Also, from equation (26) the derivative C_{n_p} appears to be most effective for airplanes with a low moment of inertia in roll or a high effective dihedral C_{l_β}.

Relation of Slope Equation to Amplitude Coefficients of Lateral Modes of Motion

The Laplace transformation (ref. 7) of the lateral equations of motion for zero initial conditions and the assumption of an input $C_n = 1$ yields the following equations:

$$\left.\begin{aligned}
&\left(2\mu_b K_Z{}^2\bar{p} - \frac{1}{2} C_{n_r}\right)\bar{p}\psi(\bar{p}) + \left(2\mu_b K_{XZ}\bar{p}^2 - \frac{1}{2} C_{n_p}\bar{p}\right)\phi(\bar{p}) - C_{n_\beta}\beta(\bar{p}) = \frac{1}{\bar{p}} \\
&\left(2\mu_b K_{XZ}\bar{p} - \frac{1}{2} C_{l_r}\right)\bar{p}\psi(\bar{p}) + \left(2\mu_b K_X{}^2\bar{p}^2 - \frac{1}{2} C_{l_p}\bar{p}\right)\phi(\bar{p}) - C_{l_\beta}\beta(\bar{p}) = 0 \\
&\left(2\mu_b - \frac{1}{2} C_{Y_r}\right)\bar{p}\psi(\bar{p}) + \left(-C_L - \frac{1}{2} C_{Y_p}\bar{p}\right)\phi(\bar{p}) + \left(2\mu_b\bar{p} - C_{Y_\beta}\right)\beta(\bar{p}) = 0
\end{aligned}\right\} \qquad (27)$$

where $\bar{p}$ is the Laplace transform variable.

By use of determinants, expressions are obtained for $\bar{p}\psi(\bar{p})$, $\phi(\bar{p})$, and $\beta(\bar{p})$ as follows:

$$\left.\begin{aligned}
\bar{p}\psi(\bar{p}) &= \frac{P_1(\bar{p})}{\bar{p}Q(\bar{p})} \\
\phi(\bar{p}) &= \frac{P_2(\bar{p})}{\bar{p}Q(\bar{p})} \\
\beta(\bar{p}) &= \frac{P_3(\bar{p})}{\bar{p}Q(\bar{p})}
\end{aligned}\right\} \qquad (28)$$

where P_1, P_2, P_3, and Q are functions of the mass and aerodynamic characteristics of the airplane. By use of the Heaviside expansion theorem (ref. 7) the inverse transformation of equations (28) yields the solutions $D\psi$, ϕ, and β as functions of the nondimensional time parameter s_b, that is,

$$\left.\begin{aligned}
D\psi &= \frac{P_1(o)}{Q(o)} + \sum_{n=1}^{m} \frac{P_1(\lambda_n)}{\lambda_n Q'(\lambda_n)} e^{\lambda_n s_b} \\
D\psi &= a_o + a_1 e^{\lambda_1 s_b} + \ldots + a_m e^{\lambda_m s_b} \\
\phi &= \frac{P_2(o)}{Q(o)} + \sum_{n=1}^{m} \frac{P_2(\lambda_n)}{\lambda_n Q'(\lambda_n)} e^{\lambda_n s_b} \\
\phi &= b_o + b_1 e^{\lambda_1 s_b} + \ldots + b_m e^{\lambda_m s_b} \\
\beta &= \frac{P_3(o)}{Q(o)} + \sum_{n=1}^{m} \frac{P_3(\lambda_n)}{\lambda_n Q'(\lambda_n)} e^{\lambda_n s_b} \\
\beta &= c_o + c_1 e^{\lambda_1 s_b} + \ldots + c_m e^{\lambda_m s_b}
\end{aligned}\right\} \qquad (29)$$

The linear and distinct roots of the lateral-stability quartic $Q(\overline{p}) = 0$ are $\lambda_1, \lambda_2, \ldots \lambda_m$, and Q' denotes differentiation with respect to $\overline{p}$. The roots of $Q(\overline{p}) = 0$ are, of course, identical with the roots discussed previously for equation (1).

The following relationships can be shown to exist:

$$\left.\begin{aligned}
\frac{\partial\lambda_n}{\partial C_{n_r}} &\equiv \frac{1}{2}\lambda_n a_n \\
\frac{\partial\lambda_n}{\partial C_{n_p}} &\equiv \frac{\lambda_n{}^2 b_n}{2} \\
\frac{\partial\lambda_n}{\partial C_{n_\beta}} &\equiv \lambda_n c_n \\
\left(\frac{\phi}{\beta}\right)_{\lambda=\lambda_n} &\equiv \frac{2}{\lambda_n}\,\frac{\partial\lambda_n/\partial C_{n_p}}{\partial\lambda_n/\partial C_{n_\beta}} \\
\left(\frac{\phi}{\psi}\right)_{\lambda=\lambda_n} &\equiv \frac{\partial\lambda_n/\partial C_{n_p}}{\partial\lambda_n/\partial C_{n_r}} \\
\left(\frac{\beta}{\psi}\right)_{\lambda=\lambda_n} &\equiv \frac{\lambda_n}{2}\,\frac{\partial\lambda_n/\partial C_{n_\beta}}{\partial\lambda_n/\partial C_{n_r}}
\end{aligned}\right\} \tag{30}$$

From the expression for ϕ/ψ, the conclusion can be made that, for airplanes with a high ϕ/ψ ratio in the Dutch roll, the Dutch roll damping will be more sensitive to the derivative C_{n_p} than to the derivative C_{n_r}.

Comparable relationships can be derived for the derivatives C_{l_r}, C_{l_p}, C_{l_β}, and C_{Y_β} for inputs $C_l = 1$ and $C_Y = 1$. For $C_l = 1$,

$$\left.\begin{aligned}
D\psi &= d_o + \sum_{n=1}^{m} d_n e^{\lambda_n s_b} \\
\phi &= e_o + \sum_{n=1}^{m} e_n e^{\lambda_n s_b} \\
\beta &= f_o + \sum_{n=1}^{m} f_n e^{\lambda_n s_b}
\end{aligned}\right\} \tag{31}$$

and therefore

$$\left.\begin{aligned}
\frac{\partial\lambda_n}{\partial C_{l_r}} &\equiv \frac{1}{2}\lambda_n d_n \\
\frac{\partial\lambda_n}{\partial C_{l_p}} &\equiv \frac{\lambda_n{}^2 e_n}{2} \\
\frac{\partial\lambda_n}{\partial C_{l_\beta}} &\equiv \lambda_n f_n \\
\left(\frac{\phi}{\beta}\right)_{\lambda=\lambda_n} &\equiv \frac{2}{\lambda_n}\frac{\partial\lambda_n/\partial C_{l_p}}{\partial\lambda_n/\partial C_{l_\beta}} \\
\left(\frac{\phi}{\psi}\right)_{\lambda=\lambda_n} &\equiv \frac{\partial\lambda_n/\partial C_{l_p}}{\partial\lambda_n/\partial C_{l_r}} \\
\left(\frac{\beta}{\psi}\right)_{\lambda=\lambda_n} &\equiv \frac{\lambda_n}{2}\frac{\partial\lambda_n/\partial C_{l_\beta}}{\partial\lambda_n/\partial C_{l_r}}
\end{aligned}\right\} \tag{32}$$

For $C_Y = 1$,

$$\left.\begin{aligned} \beta &= g_o + \sum_{n=1}^{m} g_n e^{\lambda_n s_b} \\ \phi &= h_o + \sum_{n=1}^{m} h_n e^{\lambda_n s_b} \\ D\psi &= i_o + \sum_{n=1}^{m} i_n e^{\lambda_n s_b} \end{aligned}\right\} \tag{33}$$

and therefore

$$\left.\begin{aligned} \frac{\partial\lambda_n}{\partial C_{Y_\beta}} &\equiv \lambda_n g_n \\ \frac{\partial\lambda_n}{\partial C_{Y_p}} &\equiv \frac{1}{2}\lambda_n^2 h_n \\ \frac{\partial\lambda_n}{\partial C_{Y_r}} &\equiv \frac{1}{2}\lambda_n i_n \\ \left(\frac{\phi}{\beta}\right)_{\lambda=\lambda_n} &\equiv \frac{2}{\lambda_n}\frac{\partial\lambda_n/\partial C_{Y_p}}{\partial\lambda_n/\partial C_{Y_\beta}} \\ \left(\frac{\phi}{\psi}\right)_{\lambda=\lambda_n} &\equiv \frac{\partial\lambda_n/\partial C_{Y_p}}{\partial\lambda_n/\partial C_{Y_r}} \\ \left(\frac{\beta}{\psi}\right)_{\lambda=\lambda_n} &\equiv \frac{\lambda_n}{2}\frac{\partial\lambda_n/\partial C_{Y_\beta}}{\partial\lambda_n/\partial C_{Y_r}} \end{aligned}\right\} \tag{34}$$

The coefficients a_o, b_o, c_o, d_o, e_o, f_o, g_o, h_o, and i_o which appear in equations (29), (31), and (33) are defined as follows:

$$\left.\begin{aligned}
a_o &= \frac{-2C_{l_\beta}}{C_{n_r}C_{l_\beta} - C_{l_r}C_{n_\beta}} \\
b_o &= \frac{-\left[C_{l_r}C_{Y_\beta} + \left(4\mu_b - C_{Y_r}\right)C_{l_\beta}\right]}{C_L\left(C_{n_r}C_{l_\beta} - C_{l_r}C_{n_\beta}\right)} \\
c_o &= \frac{C_{l_r}}{C_{n_r}C_{l_\beta} - C_{l_r}C_{n_\beta}} \\
d_o &= \frac{2C_{n_\beta}}{C_{n_r}C_{l_\beta} - C_{l_r}C_{n_\beta}} \\
e_o &= \frac{C_{n_r}C_{Y_\beta} + C_{n_\beta}\left(4\mu_b - C_{Y_r}\right)}{C_L\left(C_{n_r}C_{l_\beta} - C_{l_r}C_{n_\beta}\right)} \\
f_o &= \frac{-C_{n_r}}{C_{n_r}C_{l_\beta} - C_{l_r}C_{n_\beta}} \\
g_o &= 0 \\
h_o &= -\frac{1}{C_L} \\
i_o &= 0
\end{aligned}\right\} \qquad (35)$$

The following relationships exist for K_Z^2, K_X^2, and K_{XZ}:

$$\left.\begin{aligned}
\frac{\partial\lambda_n}{\partial K_Z^2} &= -4\mu_b\lambda_n \frac{\partial\lambda_n}{\partial C_{n_r}} \\
\frac{\partial\lambda_n}{\partial K_X^2} &= -4\mu_b\lambda_n \frac{\partial\lambda_n}{\partial C_{l_p}} \\
\frac{\partial\lambda_n}{\partial K_{XZ}} &= -4\mu_b\lambda_n\left(\frac{\partial\lambda_n}{\partial C_{n_p}} + \frac{\partial\lambda_n}{\partial C_{l_r}}\right)
\end{aligned}\right\} \tag{36}$$

These partial derivatives were not discussed previously but are related to $\frac{\partial\lambda_n}{\partial K_{Z_o}^2}$, $\frac{\partial\lambda_n}{\partial K_{X_o}^2}$, and $\frac{\partial\lambda_n}{\partial\eta}$ as follows:

$$\left.\begin{aligned}
\frac{\partial\lambda_n}{\partial K_{Z_o}^2} &= \frac{\partial\lambda_n}{\partial K_Z^2}\frac{\partial K_Z^2}{\partial K_{Z_o}^2} + \frac{\partial\lambda_n}{\partial K_X^2}\frac{\partial K_X^2}{\partial K_{Z_o}^2} + \frac{\partial\lambda_n}{\partial K_{XZ}}\frac{\partial K_{XZ}}{\partial K_{Z_o}^2} \\
\frac{\partial\lambda_n}{\partial K_{X_o}^2} &= \frac{\partial\lambda_n}{\partial K_Z^2}\frac{\partial K_Z^2}{\partial K_{X_o}^2} + \frac{\partial\lambda_n}{\partial K_X^2}\frac{\partial K_X^2}{\partial K_{X_o}^2} + \frac{\partial\lambda_n}{\partial K_{XZ}}\frac{\partial K_{XZ}}{\partial K_{X_o}^2} \\
\frac{\partial\lambda_n}{\partial\eta} &= \frac{\partial\lambda_n}{\partial K_Z^2}\frac{\partial K_Z^2}{\partial\eta} + \frac{\partial\lambda_n}{\partial K_X^2}\frac{\partial K_X^2}{\partial\eta} + \frac{\partial\lambda_n}{\partial K_{XZ}}\frac{\partial K_{XZ}}{\partial\eta}
\end{aligned}\right\} \tag{37}$$

Substitution of the following relationships:

$$\frac{\partial K_Z{}^2}{\partial K_{Z_o}{}^2} = \cos^2\eta \qquad \frac{\partial K_X{}^2}{\partial K_{Z_o}{}^2} = \sin^2\eta \qquad \frac{\partial K_{XZ}}{\partial K_{Z_o}{}^2} = \sin\eta\cos\eta$$

$$\frac{\partial K_Z{}^2}{\partial K_{X_o}{}^2} = \sin^2\eta \qquad \frac{\partial K_X{}^2}{\partial K_{X_o}{}^2} = \cos^2\eta \qquad \frac{\partial K_{XZ}}{\partial K_{X_o}{}^2} = -\sin\eta\cos\eta$$

$$\frac{\partial K_Z{}^2}{\partial\eta} = -2K_{XZ} \qquad \frac{\partial K_X{}^2}{\partial\eta} = 2K_{XZ} \qquad \frac{\partial K_{XZ}}{\partial\eta} = K_Z{}^2 - K_X{}^2$$

and those of equations (36) into equations (37) yields

$$\left.\begin{aligned}
\frac{\partial\lambda_n}{\partial K_{Z_o}{}^2} &= -4\mu_b\lambda_n\left[\frac{\partial\lambda_n}{\partial C_{n_r}}\cos^2\eta + \frac{\partial\lambda_n}{\partial C_{l_p}}\sin^2\eta + \left(\frac{\partial\lambda_n}{\partial C_{n_p}} + \frac{\partial\lambda_n}{\partial C_{l_r}}\right)\sin\eta\cos\eta\right] \\
\frac{\partial\lambda_n}{\partial K_{X_o}{}^2} &= -4\mu_b\lambda_n\left[\frac{\partial\lambda_n}{\partial C_{n_r}}\sin^2\eta + \frac{\partial\lambda_n}{\partial C_{l_p}}\cos^2\eta - \left(\frac{\partial\lambda_n}{\partial C_{n_p}} + \frac{\partial\lambda_n}{\partial C_{l_r}}\right)\sin\eta\cos\eta\right] \\
\frac{\partial\lambda_n}{\partial\eta} &= -4\mu_b\lambda_n\left[2K_{XZ}\left(\frac{\partial\lambda_n}{\partial C_{l_p}} - \frac{\partial\lambda_n}{\partial C_{n_r}}\right) + \left(\frac{\partial\lambda_n}{\partial C_{n_p}} + \frac{\partial\lambda_n}{\partial C_{l_r}}\right)\left(K_Z{}^2 - K_X{}^2\right)\right]
\end{aligned}\right\} \quad (38)$$

From examination of equations (30), (32), (34), (35), and (38), it becomes apparent that, for a given flight condition, calculation of the rate of change of λ_n with respect to five parameters (say, C_{n_r}, C_{n_p}, C_{n_β}, C_{l_β}, and C_{Y_β}) allows determination of the remaining partial derivatives, the amplitude coefficients associated with the lateral modes in the transient responses to unit yawing-moment, rolling-moment, or side-force inputs, and also the ratios which exist between rolling, yawing, and sideslipping in each lateral mode.

Although the preceding analysis assumed unit moments or forces as inputs, the same type of approach can be used for arbitrary inputs with similar results.

The Slope Expression as a Possible Indication of Suitable Automatic Stabilizing Devices for Airplanes

Flight tests and analysis of many of the present-day airplanes have indicated that the lateral or Dutch roll oscillation of these airplanes is very poorly damped. Usually, in order to improve this undesirable stability characteristic, artificial damping is supplied to the airplane by some type of automatic stabilization. However, the problem often exists that, although a specific stabilizer has proven to be satisfactory for one airplane, it has been found to be unsuitable for another. For instance, the damping characteristics of a certain airplane may be markedly improved by an autopilot that supplies a yawing moment to the airplane proportional to the rate of yaw but not much affected by an autopilot that supplies a yawing moment proportional to the rate of roll, and yet for another airplane the converse may be true. Also, this difference in sensitivity of airplanes to various types of artificial dampers can exist even though they possess the same or nearly the same natural period and basic damping characteristics. Since the slopes or rates of change of the roots are a measure of the sensitivity of the damping of the airplane in the different modes of motion to various parameters, they should afford some insight into the conditions that might exist in order for some of the above situations to result.

As an example, assume four airplanes which, for the same flight condition, have the same damping characteristics (same roots of characteristic equation) but differ in some of their mass and aerodynamic parameters. The characteristics of the assumed airplanes are given in the following table:

Quantity	Airplane 1	Airplane 2	Airplane 3	Airplane 4
b, ft	35.3	35.3	35.3	35.3
V, ft/sec	695.5	695.5	695.5	695.5
γ, deg	0	0	0	0
C_L	0.24	0.12	0.12	0.06
μ_b	50	50	50	50
K_X^2	0.01485	0.007425	0.01485	0.007425
K_Z^2	0.0504	0.1008	0.0504	0.1008
K_{XZ}	0	0	0	0
η, deg	0	0	0	0
C_{l_p}, per radian	-0.45	-0.225	-0.45	-0.225
C_{l_r}, per radian	0.04	0.04	0.08	0.08
C_{n_p}, per radian	-0.01	-0.01	-0.005	-0.005
C_{n_r}, per radian	-0.15	-0.30	-0.15	-0.30
C_{Y_p}, per radian	0	0	0	0
C_{Y_r}, per radian	0	0	0	0
C_{Y_β}, per radian	-0.58	-0.58	-0.58	-0.58
C_{n_β}, per radian	0.12	0.24	0.12	0.24
C_{l_β}, per radian	-0.11	-0.11	-0.22	-0.22

Although the roots of the characteristic quartic for these airplanes are the same, their sensitivity to certain parameters is different. Examination of the slopes should indicate then not only the parameters that are most effective in increasing the damping of the Dutch roll oscillation but also the reason that some parameters are more effective for one airplane than they are for another. Since the quartics for the airplanes have the same roots, the coefficients of the quartics must also be identical. Therefore, if the slopes are to differ, the reason must be a difference in the partials of the coefficients of the stability quartic with respect to the various parameters. (See eq. (4).)

Now, assume that it is desired to equip each airplane with an auxiliary damping device in order to improve the Dutch roll damping. If the dampers are assumed to have no lags, the effect of the dampers can be considered merely as a modification of the stability parameters and the slopes will present a measure of the sensitivity of the Dutch roll damping to these parameters.

The values for the rate of change of Dutch roll damping with respect to C_{n_r}, C_{n_p}, and C_{l_p} for the four airplanes are given in the following table:

Airplane	$\frac{\partial a}{\partial C_{n_r}}$	$\frac{\partial a}{\partial C_{n_p}}$	$\frac{\partial a}{\partial C_{l_p}}$
1	0.048	-0.076	0.0058
2	.024	-.076	.012
3	.048	-.15	.0058
4	.024	-.15	.012

Assume, for example, that airplanes 1 and 3 equipped with a C_{n_r} type yaw damper are satisfactorily stable. For airplanes 2 and 4, the same yaw damper would be only about 50 percent as effective as for airplanes 1 and 3. For all four airplanes it appears that, based on the slopes, a C_{n_p} damper would be more effective than a C_{n_r} damper, but, in selection of a suitable damper, consideration must be given to its effect on the other roots of the characteristic equation as well as to its effect on other factors such as the roll-to-yaw ratio. A C_{n_p} type damper, although generally effective in stabilizing the airplane Dutch roll, primarily redistributes the total damping of the system and, as a result, the damping-in-roll root λ_2 is adversely affected. Hence, the value of ΔC_{n_p} which can be tolerated before this adverse effect becomes important is usually small and the increase in Dutch roll damping is

somewhat limited. This adverse effect does not occur for the C_{n_r} type damper and, consequently, a larger increase in Dutch roll damping usually can be obtained by introducing large increments to C_{n_r}.

Nevertheless, the important point to be made here is that a difference may exist in the sensitivity of their damping to different parameters even though the airplanes initially have identical damping characteristics. Futhermore, by use of the slope equation this difference can be attributed to specific terms. For example, the reason airplanes 3 and 4 have a slope $\partial a/\partial C_{n_p}$ double that of airplanes 1 and 2 is that the values of the parameters C_{l_β} and C_{l_r} for the former airplanes are exactly double these parameters for the latter airplanes. Likewise the difference in the C_{n_r} and C_{l_p} slopes can be attributed to specific terms. It is of interest to note that the difference in $\partial a/\partial C_{n_r}$ and $\partial a/\partial C_{n_p}$ for the four airplanes will be predicted correctly by the simplified expressions (eqs. (24) and (26)).

APPLICATION OF METHOD TO THREE AIRCRAFT AND DISCUSSION OF RESULTS

Stability derivatives and mass and dimensional characteristics of three airplanes are given in table II(a). The nondimensional roots of the quartic equation (eq. (1)), the values of $1/T_{1/2}$ for the aperiodic modes, and the values of $1/T_{1/2}$ and frequency of the oscillatory mode for each airplane are presented in table II(b).

The results obtained by evaluating the exact equation (eq. (4)) $\partial\lambda/\partial x_i$ for the real roots and the complex roots for each airplane with respect to the ten parameters are given in table III. The rates of change of Dutch roll damping and frequency as calculated from equations (17) to (22) are presented in table IV along with similar data from table III as a comparison of the results obtained by utilizing these simplified equations. Generally, the agreement is very good. The slopes presented in tables III and IV are nondimensional slopes and if a comparison is made between the slopes of one airplane and those of another airplane, they must first be multiplied by the V/b ratio of the respective airplanes. These slopes are plotted in figures 1 to 4.

In a lateral-stability analysis, $T_{1/2}$ is commonly used as an indication of the degree of damping of an airplane. Here $1/T_{1/2}$ is chosen for plotting rather than $T_{1/2}$ since the former can be expressed as a linear function of Δx_i.

For the purpose of comparing the effect of increments in the parameters on the lateral stability of the three airplanes and mainly as a comparison of the approximate changes, obtained by the methods of this paper, with the exact changes, figures 1 to 4 are presented.

The straight lines in figures 1 to 4 indicate the approximate variations of the ordinate due to various parameter changes, calculated by the methods of this paper. The symbols indicate check points which were taken at equal positive and negative increments for each parameter. These points show actual variations which were obtained by increasing or decreasing the specific parameter in the lateral equations of motion and solving the characteristic quartic (eq. (1)) for the new roots. The check points are presented in the following intervals, which are the same for each figure: between $\Delta x_i/x_i$ equal to ±100 percent for C_{l_p}, C_{l_r}, and C_{n_r}; between $\Delta x_i/x_i$ equal to ±50 percent for C_{n_β}, C_{l_β}, and C_{Y_β}; between $\Delta x_i/x_i$ equal to ±20 percent for $K_{X_0}{}^2$ and $K_{Z_0}{}^2$; between ΔC_{n_p} equal to ±0.05 per radian; and between $\Delta\eta$ equal to ±6°.

Therefore, it is important to note that, with the exception of C_{n_p} and η, the variations in $1/T_{1/2}$ and ω' are not plotted against equivalent incremental scales for the three airplanes since the basic parameter x_i is generally different for each airplane. These particular intervals were not chosen as limits or boundaries for the method but were used in an attempt to show the degree to which the method could predict changes for relatively large variations of many of the parameters.

Figure 1 shows changes in the $1/T_{1/2}$ of the spiral root λ_1 of three airplanes due to increments in various airplane parameters. In figure 1(a) at $\Delta C_{l_p}/C_{l_p}$ equal to -100 percent, the actual variation has been plotted only for airplane B, since for airplanes A and C the real roots (λ_1 and λ_2) have combined to form another oscillation.

The spiral root of all the airplanes is most affected by those parameters which would be expected to cause the greatest changes as indicated from the approximate expression of the spiral root $-E/D$ where E is the constant and D the coefficient of λ in equation (1). One of the most important terms in the D coefficient is $-\mu C_{l_p} C_{n_\beta}$. As seen from figure 1, C_{l_p} and the parameters which constitute the E coefficient cause the most pronounced changes in the spiral root.

The method appears to give good approximations to the changes in the $1/T_{1/2}$ even for large variations of the parameters except for C_{l_p} and C_{n_β}. For C_{l_p}, the method predicted the changes fairly well

up to $\Delta C_{l_p}/C_{l_p}$ equal to ±40 percent. For C_{n_β}, it gave fair approximations up to $\Delta C_{n_\beta}/C_{n_\beta}$ equal to ±40 percent.

Figure 2 shows changes in the $1/T_{1/2}$ of the damping-in-roll root λ_2 of the three airplanes due to increments in various airplane parameters. The method gave good agreement to the actual changes in $1/T_{1/2}$ for all variations of the parameters considered. In figure 2(a) at $\Delta C_{l_p}/C_{l_p}$ equal to -100 percent, again the actual variation has been plotted only for airplane B. The reason for not plotting the actual variations for airplanes A and C was explained in the preceding discussion of figure 1.

A good approximation to the damping-in-roll root is $\lambda_2 \approx \frac{C_{l_p}}{4\mu_b {K_X}^2}$. Therefore, from this expression, λ_2 appears to be mainly sensitive to changes in C_{l_p} and ${K_X}^2$. Examination of figure 2 bears this out. Although ${K_X}^2$ was not plotted, the variations of this parameter will almost be equal to the variations of ${K_{X_0}}^2$.

Figure 3 shows change in $1/T_{1/2}$ of the Dutch roll oscillation of the three airplanes due to increments in various airplane parameters. The variation of Dutch roll damping with parameters other than C_{l_p} is seen to be almost linear for each of the airplanes up to the largest changes considered for the parameters, and the agreement between the calculated damping and the damping predicted by the slope equations is seen to be very good. For airplane A, however, the effect on the spiral root λ_1 (see fig. 1) and the Dutch roll damping (fig. 3(a)) of varying $\Delta C_{l_p}/C_{l_p}$ in the negative direction is predicted by the slope equations only up to $\Delta C_{l_p}/C_{l_p}$ of approximately -40 percent. The calculated points indicate a reversal of the initial effect of C_{l_p} on the Dutch roll damping beyond this value. The physical reason for this reversal is not clear. Mathematically, however, the reversal can be shown to be the result of the relationship between the variation of the total damping of the system (B/A) and the variation of the sum of λ_1 and λ_2 with C_{l_p}. This line of reasoning leads to the conclusion that the damping of the Dutch roll oscillation of an airplane will be most sensitive to variations in C_{l_p} when

$$\left|\frac{\partial(-\lambda_1 - \lambda_2)}{\partial C_{l_p}}\right| \neq \left|\frac{\partial(B/A)}{\partial C_{l_p}}\right|$$

or when these partials are equal or nearly equal and $\frac{\partial^2(-\lambda_1 - \lambda_2)}{\partial C_{l_p}{}^2}$ is large. Although the reversal of the effect of C_{l_p} on the Dutch roll damping of airplanes B and C is not apparent from figure 3(a), additional calculations indicated that this reversal does occur for larger variations in C_{l_p} than were included in the figure.

Figure 4 shows changes in the frequency of the Dutch roll oscillation of the three airplanes due to increments in various airplane parameters. The variations in frequency calculated from the slope equations are seen from these figures to agree almost identically with the actual variations for all of the parameters considered. A generally satisfactory approximation of the nondimensional frequency of the Dutch roll oscillation is

$$\omega \approx \left(\frac{C_{n_\beta}}{2\mu_b K_Z{}^2} - \frac{K_{XZ}C_{l_\beta}}{2\mu_b K_X{}^2 K_Z{}^2}\right)^{1/2}$$

From the above expression the frequency appears to be mainly sensitive to $K_Z{}^2$, C_{n_β}, $K_X{}^2$, K_{XZ}, and C_{l_β} (and μ_b which was not considered in this paper). Examination of figure 4 bears out the validity of the expression as an approximation to ω for the airplanes considered. The parameters C_{n_β} and $K_{Z_0}{}^2$ appear to have the greatest effect on ω but it should be pointed out that K_{XZ} for the airplanes was generally small and therefore the term $\frac{-K_{XZ}C_{l_\beta}}{2\mu_b K_X{}^2 K_Z{}^2}$ contributed very little to ω for these airplanes. The general effect of this term can be seen from figure 4(d). As η is assumed to be increased positively, the term becomes more positive and ω is increased somewhat.

CONCLUDING REMARKS

Expressions have been presented in this paper which afford a means of calculating the rate of change of the roots of the lateral-stability quartic equation with respect to ten mass and aerodynamic parameters upon which these roots depend. Application of these expressions to three airplanes indicated that satisfactory results are obtained for

large variations in the ten parameters even though in the derivation of the expressions small variations in the parameters were assumed. An attempt was made to simplify certain of the expressions and the results indicated good agreement when compared with the results of the exact expressions.

The expressions which define the rate of change of the characteristic stability roots with respect to the various parameters were shown to bear a definite relationship with the amplitude coefficients (and hence the ratios) of the lateral modes of motion subsequent to applied moments and forces. From these relationships calculation of the rate of change of λ_n with respect to five prescribed parameters allows determination of the remaining partial derivatives and these amplitude coefficients and ratios.

The slope expressions should afford some insight into the types of automatic stabilization suitable for a given airplane since, if the dynamics of the automatic stabilizer are assumed to be negligible, the automatic stabilizer is effectively varying one or more of the mass or aerodynamic parameters of the airplane.

The effect of a parameter change on a given airplane may be different from the effect of the same parameter change on another airplane. The reasons for this difference can be seen from examination of the slope equation and, in some instances, can be attributed to particular parameters or combination of parameters. For example, the effect of the damping-in-yaw stability derivative C_{n_r} appears to depend primarily on the moment of inertia in yaw, whereas the effect of the yawing-moment coefficient due to rolling velocity C_{n_p} tends to be most effective for airplanes with low moment of inertia in roll or high effective dihedral. The relative effectiveness of C_{n_p} and C_{n_r} can be measured by the roll-to-yaw ratio ϕ/ψ of a given airplane, and for high values of ϕ/ψ, C_{n_p} is the most effective of the two.

A method for calculating approximately the roots of the stability characteristic quartic is presented in the appendix and is shown to give very accurate results for the three airplanes considered.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., October 2, 1953.

APPENDIX A

APPROXIMATE SOLUTION OF LATERAL STABILITY QUARTIC

The period and damping of the various lateral modes of motion are determined from the linear and distinct roots of an equation of the form

$$F(\lambda) = A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \tag{A1}$$

where the coefficients A, B, C, D, and E are functions of the stability derivatives and mass parameters of the airplane. Rewriting, after division by A, gives

$$F(\lambda) = \lambda^4 + B'\lambda^3 + C'\lambda^2 + D'\lambda + E' = 0 \tag{A2}$$

Assume that

$$(\lambda^2 + p_1\lambda + q_1)(\lambda^2 + p_2\lambda + q_2) \equiv F(\lambda)$$

Hence,

$$\left.\begin{aligned} B' &= p_1 + p_2 \\ C' &= q_1 + q_2 + p_1p_2 \\ D' &= p_1q_2 + p_2q_1 \\ E' &= q_1q_2 \end{aligned}\right\} \tag{A3}$$

Equation (A2) will almost always have two real roots and a pair of conjugate complex roots. One of the real roots λ_1 is associated with the inherently very poor spiral stability of airplanes and hence is very small. The remaining real root λ_2 is associated with the damping in roll of the airplane and is of the order of the coefficient B'. The real and imaginary parts of the complex roots determine the damping and period, respectively, of the normal lateral Dutch roll oscillation. If the assumption is made that the complex roots are obtained from the

quadratic $\lambda^2 + p_1\lambda + q_1$, the condition generally exists that $q_2 \ll q_1$, and hence q_2 can be assumed to be zero in equations (A3). Therefore,

$$\left.\begin{aligned} B' &= p_1 + p_2 \\ C' &= q_1 + p_1 p_2 \\ D' &= p_2 q_1 \\ E' &= 0 \end{aligned}\right\} \tag{A4}$$

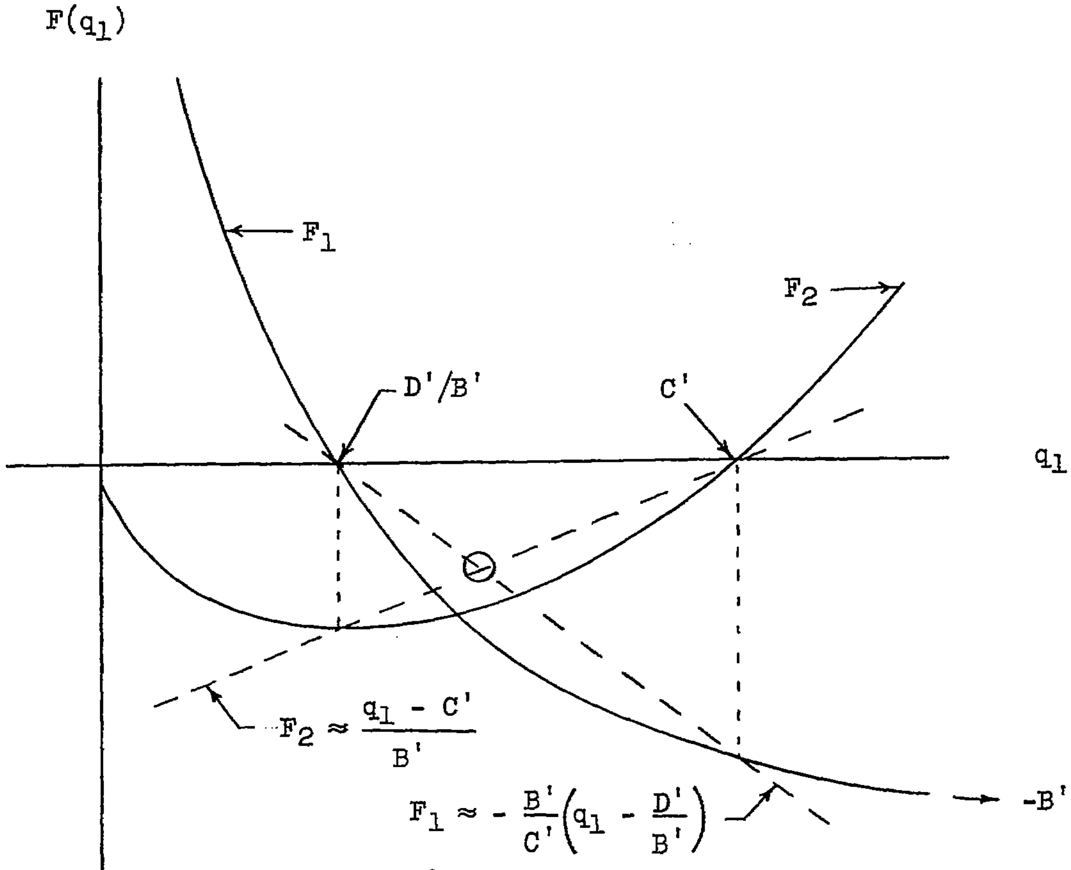
From equations (A4) the following expressions are obtained for p_1:

$$\left.\begin{aligned} p_1 &= B' - \frac{D'}{q_1} \\ p_1 &= \frac{q_1(C' - q_1)}{D'} \end{aligned}\right\} \tag{A5}$$

Equating equations (A5) yields

$$\left.\begin{aligned} \frac{D'}{q_1} - B' &= \frac{q_1(q_1 - C')}{D'} \\ F_1(q_1) &= F_2(q_1) \end{aligned}\right\} \tag{A6}$$

Therefore, q_1 can be determined by a simultaneous solution of equations (A5). The functions F_1 and F_2 are illustrated by the following sketch:



Replace F_1 and F_2 for $\frac{D'}{B'} < q_1 < C'$ by the expressions

$$\left.\begin{aligned} F_1 &\approx -\frac{B'}{C'}\left(q_1 - \frac{D'}{B'}\right) \\ F_2 &\approx \frac{q_1 - C'}{B'} \end{aligned}\right\} \tag{A7}$$

The intersection of these straight lines gives the desired q_1, which is

$$q_1 = \frac{(C')^2 + B'D'}{(B')^2 + C'} \tag{A8}$$

Substitution of this value of q_1 into the first of equations (A5) gives

$$p_1 = \frac{C'(B'C' - D')}{(C')^2 + B'D'} \tag{A9}$$

or, since $p_1 = -2a$ (where a is the real part of the complex root $\lambda = a + i\omega$),

$$a = -\frac{1}{2}\frac{C'(B'C' - D')}{(C')^2 + B'D'} \tag{A10}$$

Also, the imaginary part of the complex root is related to q_1 by

$$\omega = (q_1 - a^2)^{1/2} \tag{A11}$$

The two real roots may be obtained as follows:

From equation (A4)

$$p_2 \approx \frac{D'}{q_1}$$

and from equation (A3)

$$q_2 = \frac{E'}{q_1} \tag{A12}$$

Therefore, the quadratic giving the real roots may be written as

$$\lambda^2 + \frac{D'}{q_1}\lambda + \frac{E'}{q_1} = 0$$

Solving the above quadratic for λ gives

$$\lambda = \frac{1}{2q_1}\left\{-D' \pm \left[(D')^2 - 4q_1E'\right]^{1/2}\right\}$$

Then, the spiral root is given by the root with the plus sign

$$\lambda_1 = \frac{1}{2q_1}\left\{-D' + \left[(D')^2 - 4q_1E'\right]^{1/2}\right\} \tag{A13}$$

and the damping-in-roll root is given by the root with the minus sign

$$\lambda_2 = \frac{1}{2q_1}\left\{-D' - \left[(D')^2 - 4q_1E'\right]^{1/2}\right\} \tag{A14}$$

Hence, the four roots of equations (1) may be found by equations (A8), (A10), (A11), (A13), and (A14).

It should be reiterated that these results are contingent primarily upon the condition that $q_2 \ll q_1$ and if, upon determination of q_2 (eq. (A12)), this condition is violated, the results are, of course, invalid, and resort must be made to other methods of determining the roots of the lateral-stability quartic.

The following table shows a comparison between the values of the nondimensional roots obtained by using this approximate method and the exact roots for the three airplanes:

	Spiral root	Damping-in-roll root	Dutch roll oscillation
		Airplane A	
Exact Approximate	-0.0004107 -.0004099	-0.13932 -.13947	-0.0094337 ± 0.17271 -.009330 ± .17121
		Airplane B	
Exact Approximate	-0.0007611 -.0007646	-0.036142 -.03598	-0.00004245 ± 0.0709111 -.0001658 ± .071061
		Airplane C	
Exact Approximate	-0.00049 -.00049	-0.15679 -.15656	-0.00746 ± 0.156731 -.00757 ± .156861

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TABLE I.- GENERAL FORMS OF THE PARTIALS OF THE QUARTIC COEFFICIENTS FOR TEN DIFFERENT PARAMETERS

C_{n_r}	
$\partial A/\partial C_{n_r}$	0
$\partial B/\partial C_{n_r}$	$-2\mu_b^2 K_X^2$
$\partial C/\partial C_{n_r}$	$\mu_b\left(K_X^2 C_{Y_\beta} + \frac{1}{2} C_{l_p}\right)$
$\partial D/\partial C_{n_r}$	$\frac{1}{4}\left(C_{l_\beta} C_{Y_p} - C_{l_p} C_{Y_\beta}\right)$
$\partial E/\partial C_{n_r}$	$\frac{1}{2} C_L C_{l_\beta}$
C_{n_p}	
$\partial A/\partial C_{n_p}$	0
$\partial B/\partial C_{n_p}$	$2\mu_b^2 K_{XZ}$
$\partial C/\partial C_{n_p}$	$-\mu_b\left(K_{XZ} C_{Y_\beta} + \frac{1}{2} C_{l_r}\right)$
$\partial D/\partial C_{n_p}$	$\frac{1}{4} C_{l_r} C_{Y_\beta} + \mu_b C_{l_\beta} - \frac{1}{4} C_{l_\beta} C_{Y_r}$
$\partial E/\partial C_{n_p}$	0
C_{l_r}	
$\partial A/\partial C_{l_r}$	0
$\partial B/\partial C_{l_r}$	$2\mu_b^2 K_{XZ}$
$\partial C/\partial C_{l_r}$	$-\mu_b\left(K_{XZ} C_{Y_\beta} + \frac{1}{2} C_{n_p}\right)$
$\partial D/\partial C_{l_r}$	$\frac{1}{4}\left(C_{n_p} C_{Y_\beta} - C_{n_\beta} C_{Y_p}\right)$
$\partial E/\partial C_{l_r}$	$-\frac{1}{2} C_L C_{n_\beta}$

TABLE I.- GENERAL FORMS OF THE PARTIALS OF THE QUARTIC COEFFICIENTS FOR TEN DIFFERENT PARAMETERS - Continued

C_{l_p}	
$\partial A/\partial C_{l_p}$	0
$\partial B/\partial C_{l_p}$	$-2\mu_b{}^2K_Z{}^2$
$\partial C/\partial C_{l_p}$	$\mu_b\left(K_Z{}^2C_{Y_\beta} + \frac{1}{2}C_{n_r}\right)$
$\partial D/\partial C_{l_p}$	$-\frac{1}{4}C_{n_r}C_{Y_\beta} - \mu_bC_{n_\beta} + \frac{1}{4}C_{n_\beta}C_{Y_r}$
$\partial E/\partial C_{l_p}$	0
C_{Y_β}	
$\partial A/\partial C_{Y_\beta}$	0
$\partial B/\partial C_{Y_\beta}$	$-4\mu_b{}^2\left(K_X{}^2K_Z{}^2 - K_{XZ}{}^2\right)$
$\partial C/\partial C_{Y_\beta}$	$\mu_b\left(K_X{}^2C_{n_r} + K_Z{}^2C_{l_p} - K_{XZ}C_{l_r} - K_{XZ}C_{n_p}\right)$
$\partial D/\partial C_{Y_\beta}$	$\frac{1}{4}\left(C_{n_p}C_{l_r} - C_{n_r}C_{l_p}\right)$
$\partial E/\partial C_{Y_\beta}$	0
C_{n_β}	
$\partial A/\partial C_{n_\beta}$	0
$\partial B/\partial C_{n_\beta}$	0
$\partial C/\partial C_{n_\beta}$	$\mu_b\left(4\mu_bK_X{}^2 + K_{XZ}C_{Y_p} - K_X{}^2C_{Y_r}\right)$
$\partial D/\partial C_{n_\beta}$	$-\mu_bC_{l_p} + 2\mu_bC_LK_{XZ} + \frac{1}{4}C_{l_p}C_{Y_r} - \frac{1}{4}C_{l_r}C_{Y_p}$
$\partial E/\partial C_{n_\beta}$	$-\frac{1}{2}C_LC_{l_r}$

TABLE I.- GENERAL FORMS OF THE PARTIALS OF THE QUARTIC COEFFICIENTS FOR TEN DIFFERENT PARAMETERS - Concluded

	C_{l_β}
$\partial A/\partial C_{l_\beta}$	0
$\partial B/\partial C_{l_\beta}$	0
$\partial C/\partial C_{l_\beta}$	$-\mu_b\left(4\mu_b K_{XZ} + K_Z^2 C_{Y_p} - K_{XZ}C_{Y_r}\right)$
$\partial D/\partial C_{l_\beta}$	$\mu_b C_{n_p} - 2\mu_b C_L K_Z^2 - \frac{1}{4} C_{n_p}C_{Y_r} + \frac{1}{4} C_{n_r}C_{Y_p}$
$\partial E/\partial C_{l_\beta}$	$\frac{1}{2} C_L C_{n_r}$
	${K_X}_o^2$
$\partial A/\partial {K_X}_o^2$	$8\mu_b^3 {K_Z}_o^2$
$\partial B/\partial {K_X}_o^2$	$-2\mu_b^2\left[2C_{Y_\beta}{K_Z}_o^2 + C_{n_r}\cos^2\eta + C_{l_p}\sin^2\eta + \left(C_{l_r} + C_{n_p}\right)\sin\eta\cos\eta\right]$
$\partial C/\partial {K_X}_o^2$	$\mu_b\left[\cos^2\eta\left(C_{n_r}C_{Y_\beta} + 4\mu_b C_{n_\beta} - C_{Y_r}C_{n_\beta}\right) + \sin^2\eta\left(C_{l_p}C_{Y_\beta} - C_{Y_p}C_{l_\beta}\right) + \sin\eta\cos\eta\left(C_{l_r}C_{Y_\beta} + 4\mu_b C_{l_\beta} + C_{n_p}C_{Y_\beta} - C_{n_\beta}C_{Y_p} - C_{Y_r}C_{l_\beta}\right)\right]$
$\partial D/\partial {K_X}_o^2$	$-2\mu_b C_L\left(C_{l_\beta}\sin^2\eta + C_{n_\beta}\sin\eta\cos\eta\right)$
$\partial E/\partial {K_X}_o^2$	0
	${K_Z}_o^2$
$\partial A/\partial {K_Z}_o^2$	$8\mu_b^3 {K_X}_o^2$
$\partial B/\partial {K_Z}_o^2$	$-2\mu_b^2\left[2C_{Y_\beta}{K_X}_o^2 + C_{n_r}\sin^2\eta + C_{l_p}\cos^2\eta - \left(C_{l_r} + C_{n_p}\right)\sin\eta\cos\eta\right]$
$\partial C/\partial {K_Z}_o^2$	$\mu_b\left[\sin^2\eta\left(C_{n_r}C_{Y_\beta} + 4\mu_b C_{n_\beta} - C_{Y_r}C_{n_\beta}\right) + \cos^2\eta\left(C_{l_p}C_{Y_\beta} - C_{Y_p}C_{l_\beta}\right) - \sin\eta\cos\eta\left(C_{l_r}C_{Y_\beta} + 4\mu_b C_{l_\beta} + C_{n_p}C_{Y_\beta} - C_{n_\beta}C_{Y_p} - C_{Y_r}C_{l_\beta}\right)\right]$
$\partial D/\partial {K_Z}_o^2$	$-2\mu_b C_L\left(C_{l_\beta}\cos^2\eta - C_{n_\beta}\sin\eta\cos\eta\right)$
$\partial E/\partial {K_Z}_o^2$	0
	η
$\partial A/\partial\eta$	0
$\partial B/\partial\eta$	$2\mu_b^2\left[2K_{XZ}\left(C_{l_p} - C_{n_r}\right) + \left(C_{l_r} + C_{n_p}\right)\left({K_Z}_o^2 - {K_X}_o^2\right)\cos 2\eta\right]$
$\partial C/\partial\eta$	$\mu_b\left[2K_{XZ}\left(C_{n_r}C_{Y_\beta} + 4\mu_b C_{n_\beta} - C_{l_p}C_{Y_\beta} + C_{Y_p}C_{l_\beta} - C_{Y_r}C_{n_\beta}\right) - \left({K_Z}_o^2 - {K_X}_o^2\right)\left(C_{l_r}C_{Y_\beta} + 4\mu_b C_{l_\beta} + C_{n_p}C_{Y_\beta} - C_{n_\beta}C_{Y_p} - C_{Y_r}C_{l_\beta}\right)\cos 2\eta\right]$
$\partial D/\partial\eta$	$2\mu_b C_L\left[2K_{XZ}C_{l_\beta} + C_{n_\beta}\left({K_Z}_o^2 - {K_X}_o^2\right)\cos 2\eta\right]$
$\partial E/\partial\eta$	0

TABLE II.- CHARACTERISTICS OF AIRPLANES CONSIDERED

(a) Stability derivatives and mass and dimensional characteristics of three airplanes

	Airplane A	Airplane B	Airplane C
Wing:			
Span, ft	28.00	25.00	35.30
Area, ft^2	130.00	175.00	250.00
Aspect ratio	6.00	3.60	4.97
Sweepback, deg	5.05	35.00	0
Dihedral, deg	0	-3.00	4.00
W, lb	8,450.00	9,245.00	12,630.00
W/S, lb/ft^2	65.00	52.80	50.50
Altitude, ft	30,000.00	50,000.00	30,000.00
V, ft/sec	797.00	776.00	695.00
C_L	0.23	0.49	0.24
μ_b	80.70	182.00	50.00
η, deg	-2.00	0.82	0
$K_{X_0}{}^2$	0.00962	0.01557	0.01485
$K_{Z_0}{}^2$	0.05135	0.1560	0.0504
$K_X{}^2$	0.00967	0.0156	0.01485
$K_Z{}^2$	0.0513	0.156	0.0504
K_{XZ}	-0.00145	0.0020	0
C_{l_p}, per radian	-0.40	-0.33	-0.45
C_{l_r}, per radian	0.08	0.23	0.04
C_{n_p}, per radian	-0.02	-0.05	-0.01
C_{n_r}, per radian	-0.40	-0.69	-0.15
C_{Y_p}, per radian	0	0	0
C_{Y_r}, per radian	0	0	0
C_{Y_β}, per radian	-1.0	-0.58	-0.58
C_{n_β}, per radian	0.25	0.25	0.12
C_{l_β}, per radian	-0.126	-0.18	-0.11
tan γ	0	0	0

TABLE II.- CHARACTERISTICS OF AIRPLANES CONSIDERED - Concluded

(b) Nondimensional roots of the characteristic quartic equation, with the respective values of $\left(1/T_{1/2}\right)_o$ and $(\omega')_o$ for the three airplanes

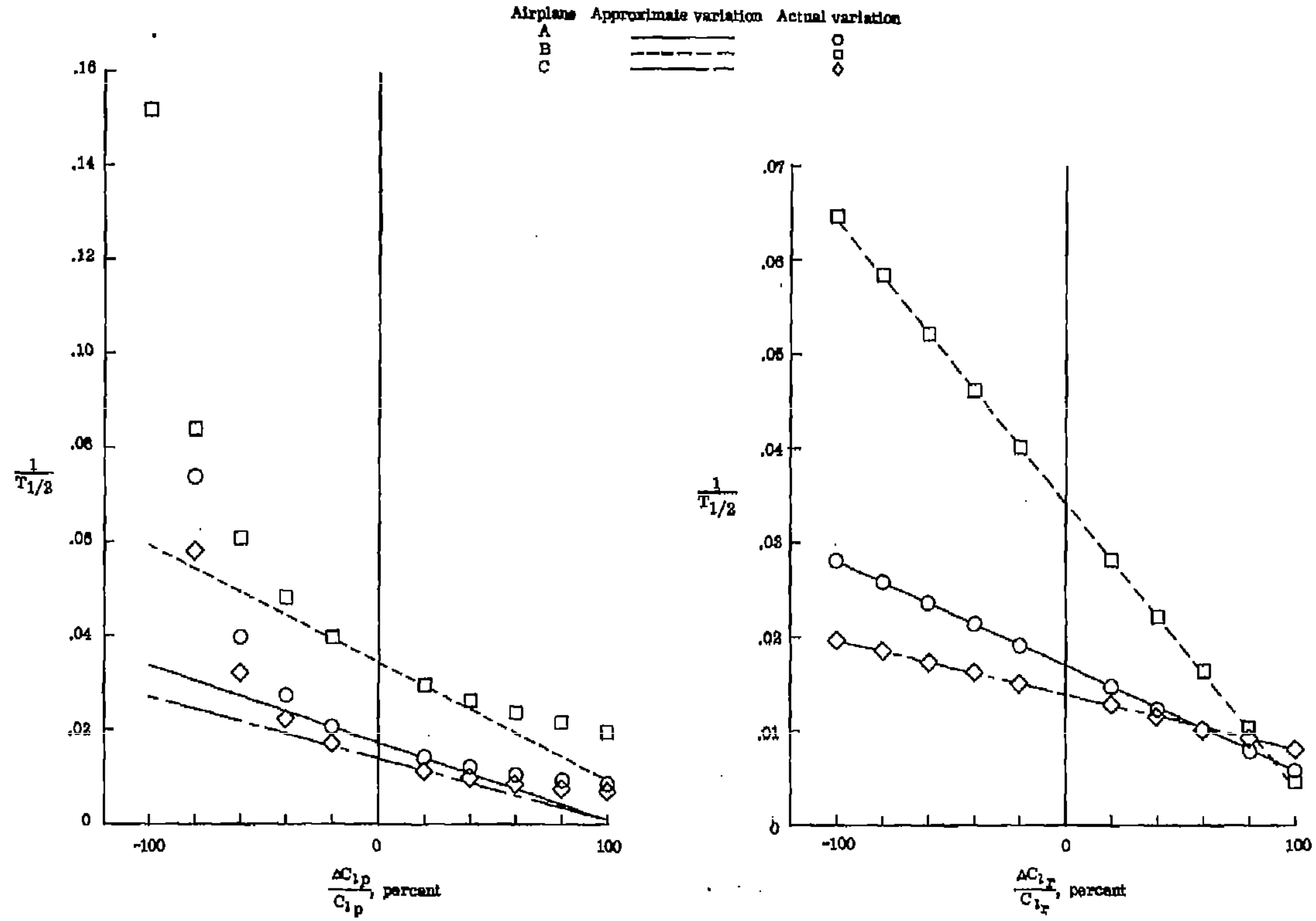
	Nondimensional roots, λ	Damping factor, $\left(1/T_{1/2}\right)_o$	Frequency, $(\omega')_o$
Airplane A:			
Spiral root	-0.0004107	0.01687	----------
Damping-in-roll root . . .	-0.13932	5.7220	----------
Dutch roll oscillation . .	-0.0094337 ± 0.171271	0.3875	4.875
Airplane B:			
Spiral root	-0.0007611	0.03409	----------
Damping-in-roll root . . .	-0.036142	1.6190	----------
Dutch roll oscillation . .	-0.00004245 ± 0.0709111	0.001901	2.201
Airplane C:			
Spiral root	-0.00049	0.01393	----------
Damping-in-roll root . . .	-0.15679	4.4580	----------
Dutch roll oscillation . .	-0.00746 ± 0.156731	0.2121	3.088

TABLE III.- NONDIMENSIONAL SLOPES FOR THREE AIRPLANES WITH RESPECT TO TEN PARAMETERS

x_i	Slopes			
	$\partial\lambda_1/\partial x_i$	$\partial\lambda_2/\partial x_i$	$\partial a/\partial x_i$	$\partial\omega/\partial x_i$
Airplane A				
C_{l_p}	-0.0010	0.32	0.0024	-0.0090
C_{l_r}	.0034	.0026	.0015	.0040
C_{n_r}	.0017	.00088	.029	.0023
C_{n_p}	-.00049	.11	-.048	-.041
C_{n_β}	.0026	.030	-.016	.34
C_{l_β}	.0052	.088	-.047	.020
C_{Y_β}	-.0000019	.00018	.0030	-.00013
η	.000017	.16	-.088	.098
${K_Z}_0^2$	.00021	-.11	.29	-1.68
${K_X}_0^2$	-.00012	14.47	-.57	-.076
Airplane B				
C_{l_p}	-0.0017	0.077	0.0062	-0.0015
C_{l_r}	.0029	-.0023	-.00086	.00057
C_{n_r}	.0021	-.0013	.0040	.00072
C_{n_p}	-.0012	.044	-.022	-.013
C_{n_β}	.0048	.018	-.011	.12
C_{l_β}	.0067	.031	-.019	-.024
C_{Y_β}	-.0000019	.000094	.0013	.0000060
η	.00013	.16	-.089	.16
${K_Z}_0^2$	.0012	-.019	.028	-.19
${K_X}_0^2$	-.00094	2.022	-.068	-.34
Airplane C				
C_{l_p}	-0.0010	0.33	0.0058	-0.00060
C_{l_r}	.0050	-.0016	-.0017	.0019
C_{n_r}	.0046	-.00074	.048	.0028
C_{n_p}	-.00094	.15	-.076	-.077
C_{n_β}	.0055	.021	-.013	.62
C_{l_β}	.0060	.046	-.026	-.022
C_{Y_β}	-.0000028	.00016	.0049	-.00013
η	.000014	.17	-.088	.083
${K_Z}_0^2$	.00045	-.023	.16	-1.49
${K_X}_0^2$	-.00010	10.22	-.010	-.18

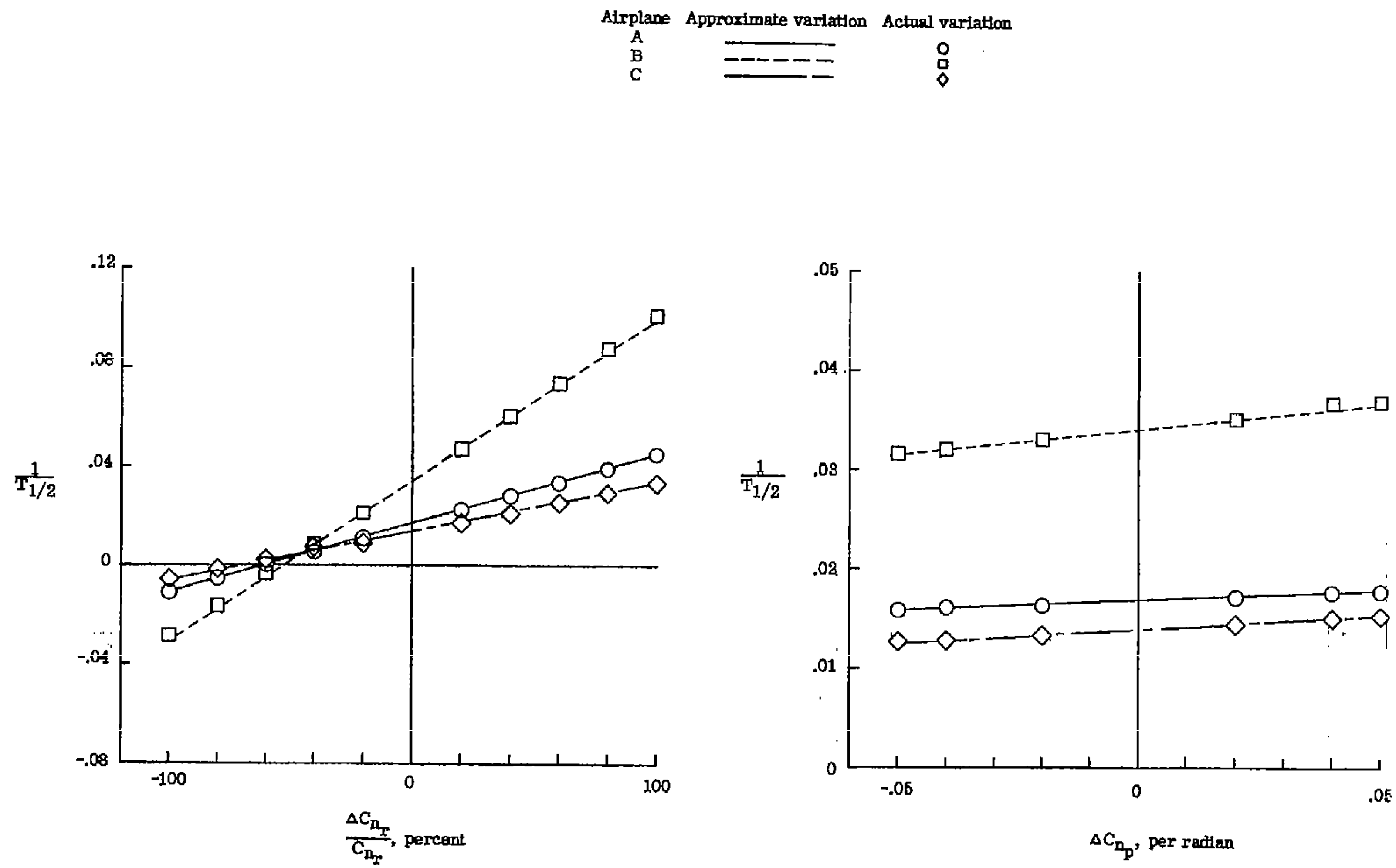
TABLE IV.- RATE OF CHANGE OF DUTCH ROLL DAMPING AND FREQUENCY OBTAINED FROM SIMPLIFIED EXPRESSIONS COMPARED WITH THE EXACT

x_i	$\partial a/\partial x_i$		$\partial \omega/\partial x_i$	
	Simplified	Exact	Simplified	Exact
Airplane A				
C_{l_p}	0.0031	0.0024	-0.0086	-0.0090
C_{l_r}	.0016	.0015	.0037	.0040
C_{n_r}	.029	.029	.0017	.0023
C_{n_p}	-.045	-.048	-.042	-.041
C_{n_β}	-.014	-.016	.34	.34
C_{l_β}	-.045	-.047	.019	.020
C_{Y_β}	.0030	.0030	-.00016	-.00013
η	-.074	-.088	.098	.098
$K_{Z_o}^2$	.29	.29	-1.69	-1.68
$K_{X_o}^2$	-.54	-.57	-.10	-.076
Airplane B				
C_{l_p}	0.0062	0.0062	-0.0016	-0.0015
C_{l_r}	-.00085	-.00086	.00058	.00057
C_{n_r}	.0040	.0040	.00067	.00072
C_{n_p}	-.022	-.022	-.012	-.013
C_{n_β}	-.0099	-.011	.12	.12
C_{l_β}	-.019	-.019	-.027	-.024
C_{Y_β}	.0013	.0013	-.000009	.000006
η	-.087	-.089	.16	.16
$K_{Z_o}^2$	.026	.028	-.19	-.19
$K_{X_o}^2$	-.071	-.068	-.33	-.34
Airplane C				
C_{l_p}	0.0058	0.0058	-0.00021	-0.00060
C_{l_r}	-.0018	-.0017	.0018	.0019
C_{n_r}	.048	.048	.0024	.0028
C_{n_p}	-.073	-.076	-.077	-.077
C_{n_β}	-.011	-.013	.62	.62
C_{l_β}	-.025	-.026	-.022	-.022
C_{Y_β}	.0049	.0049	-.00015	-.00013
η	-.083	-.088	.083	.083
$K_{Z_o}^2$	.17	.16	-1.49	-1.49
$K_{X_o}^2$	-.013	-.010	-.17	-.18



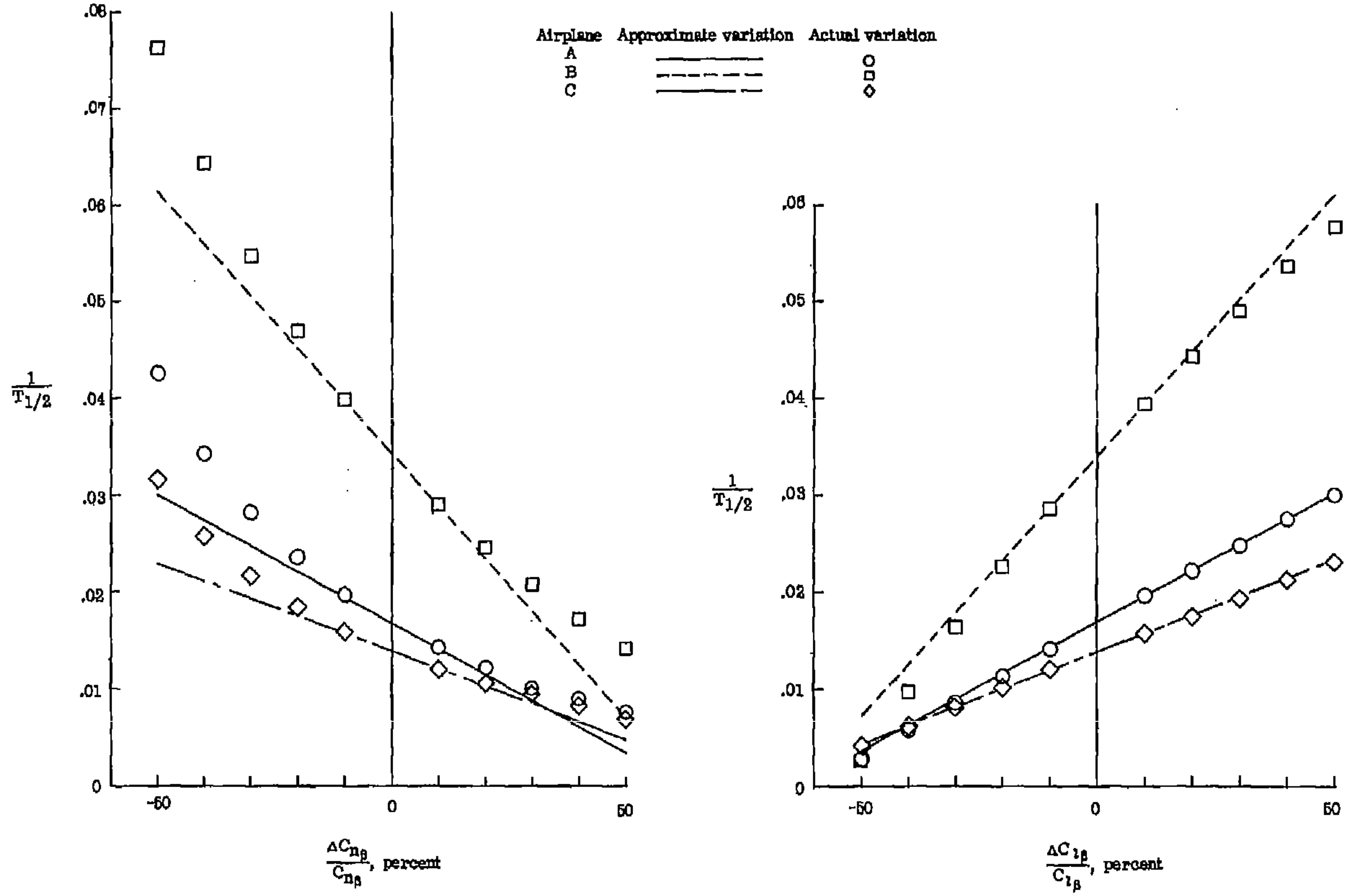
(a) C_{l_p} and C_{l_r}.

Figure 1.- Changes in the $1/T_{1/2}$ of the spiral root of three airplanes due to increments in various airplane parameters.



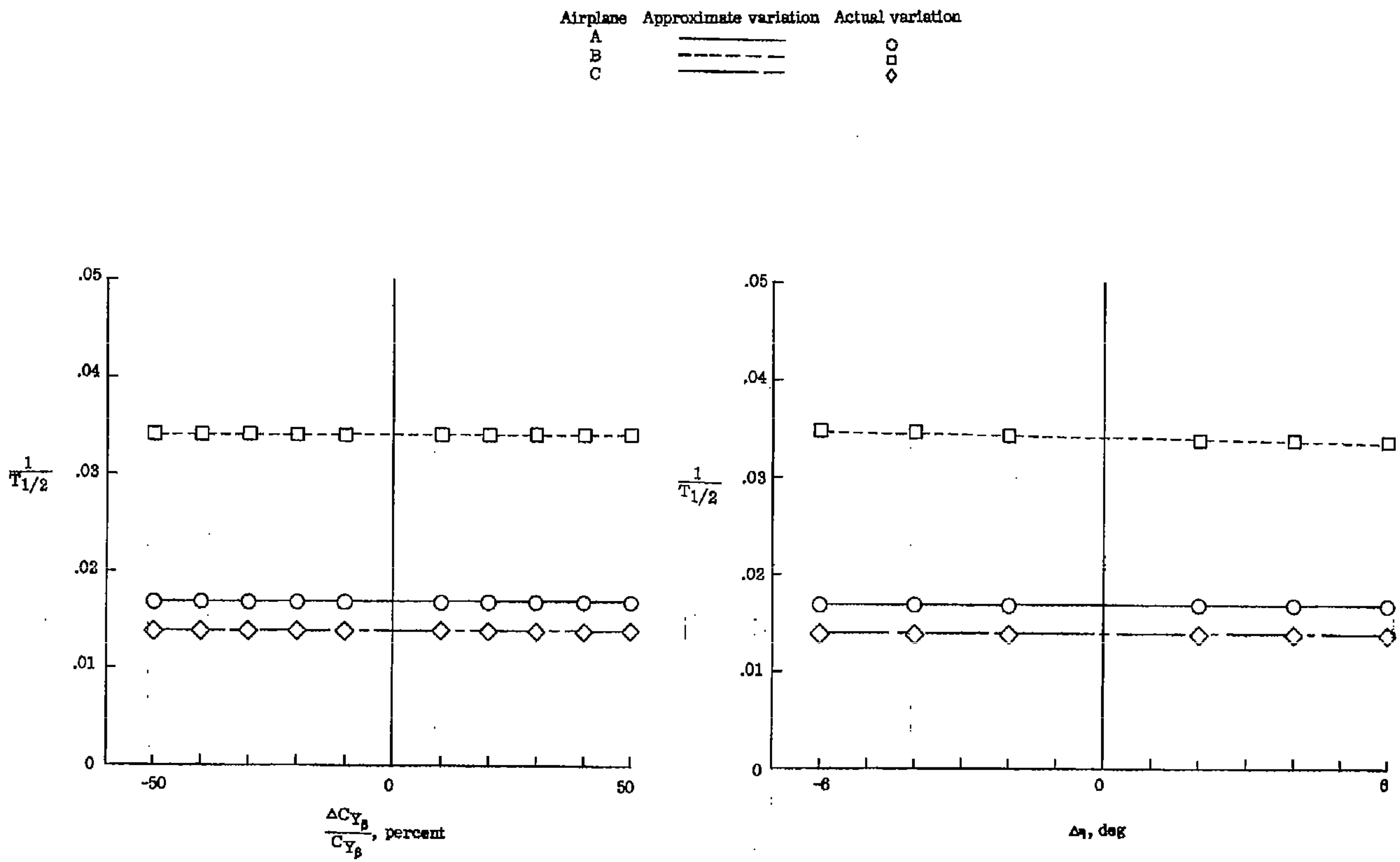
(b) C_{n_r} and C_{n_p}.

Figure 1.- Continued.



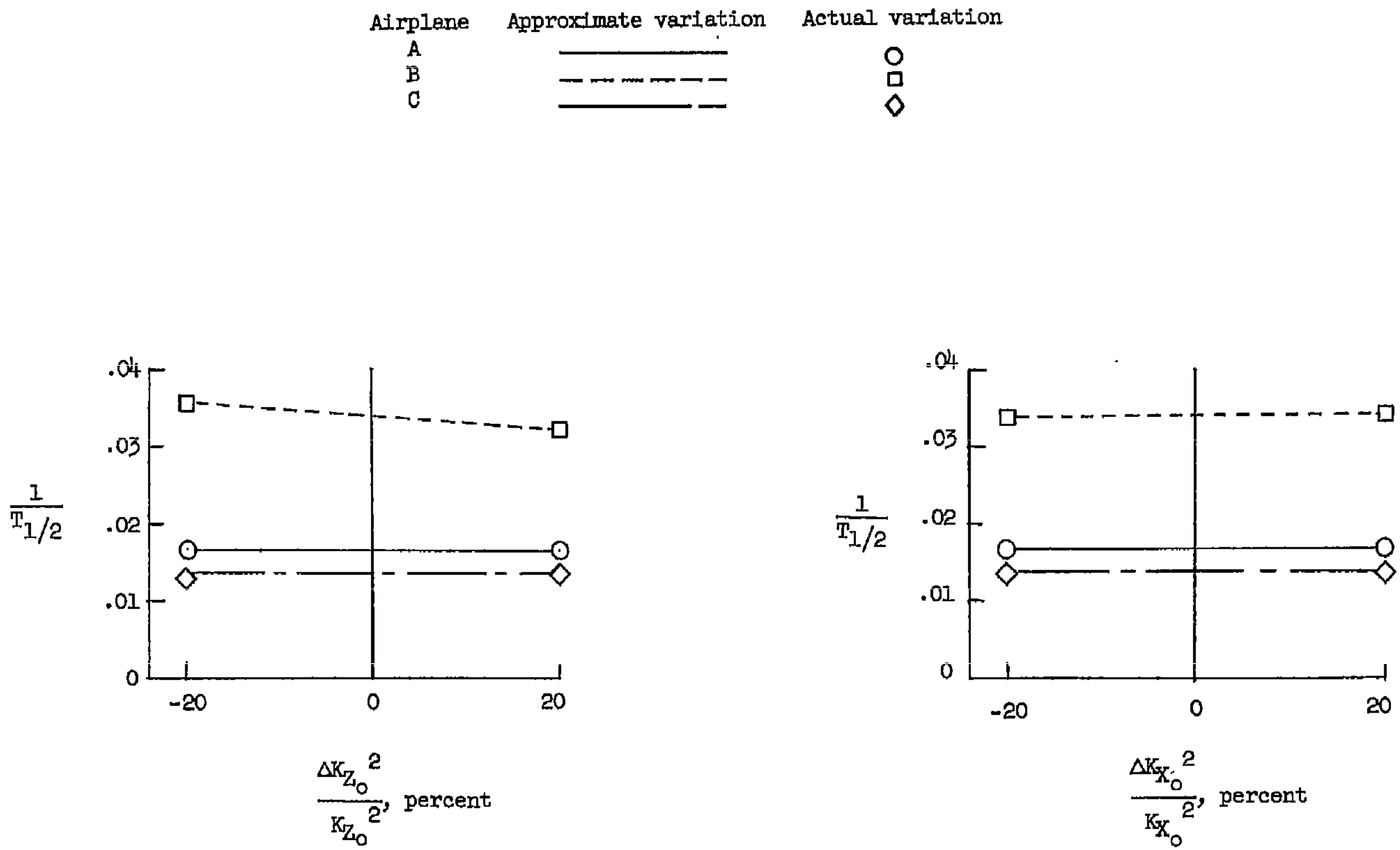
(c) C_{n_β} and C_{l_β}.

Figure 1.- Continued.



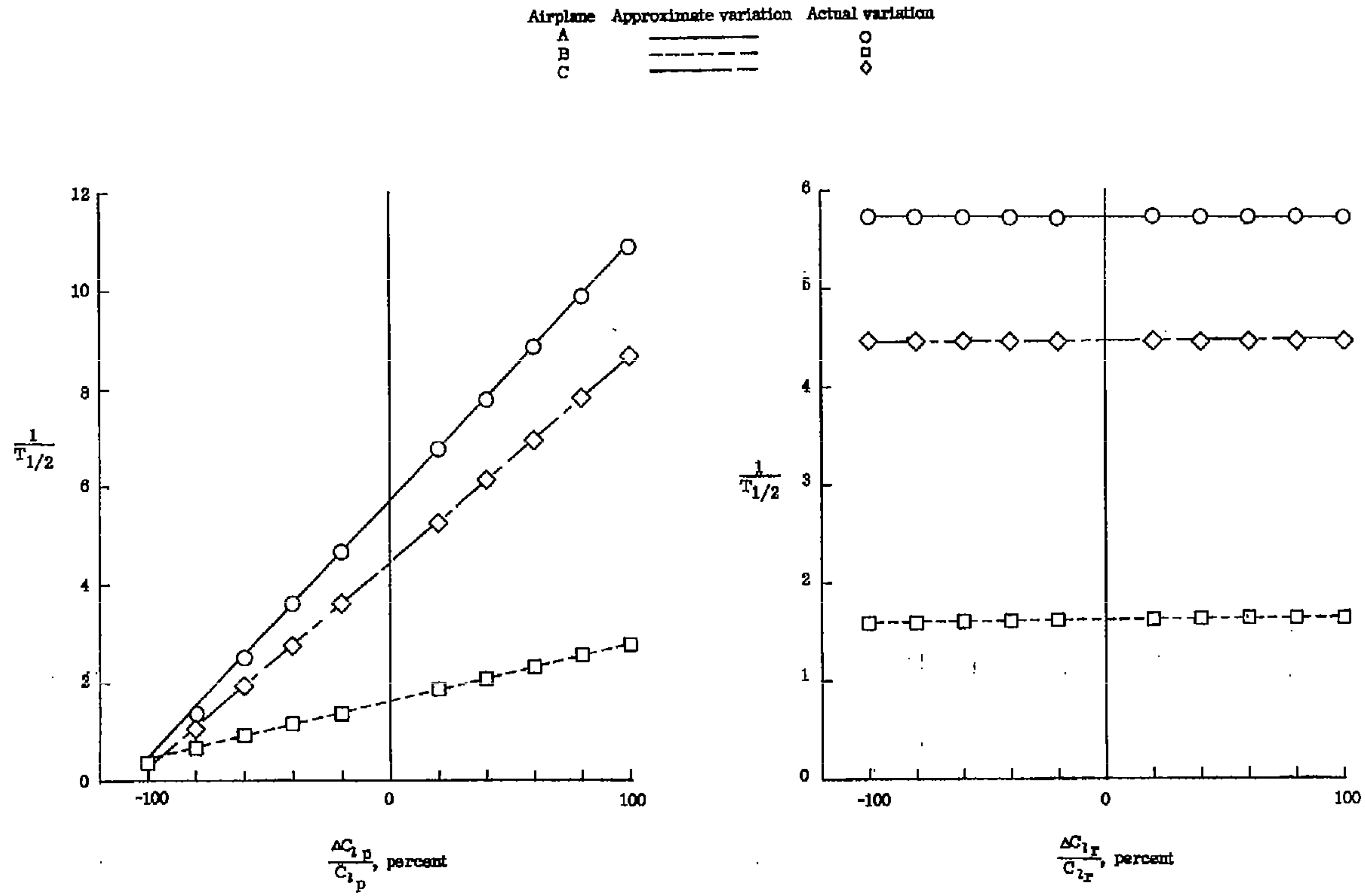
(d) C_{Y_β} and η.

Figure 1.- Continued.



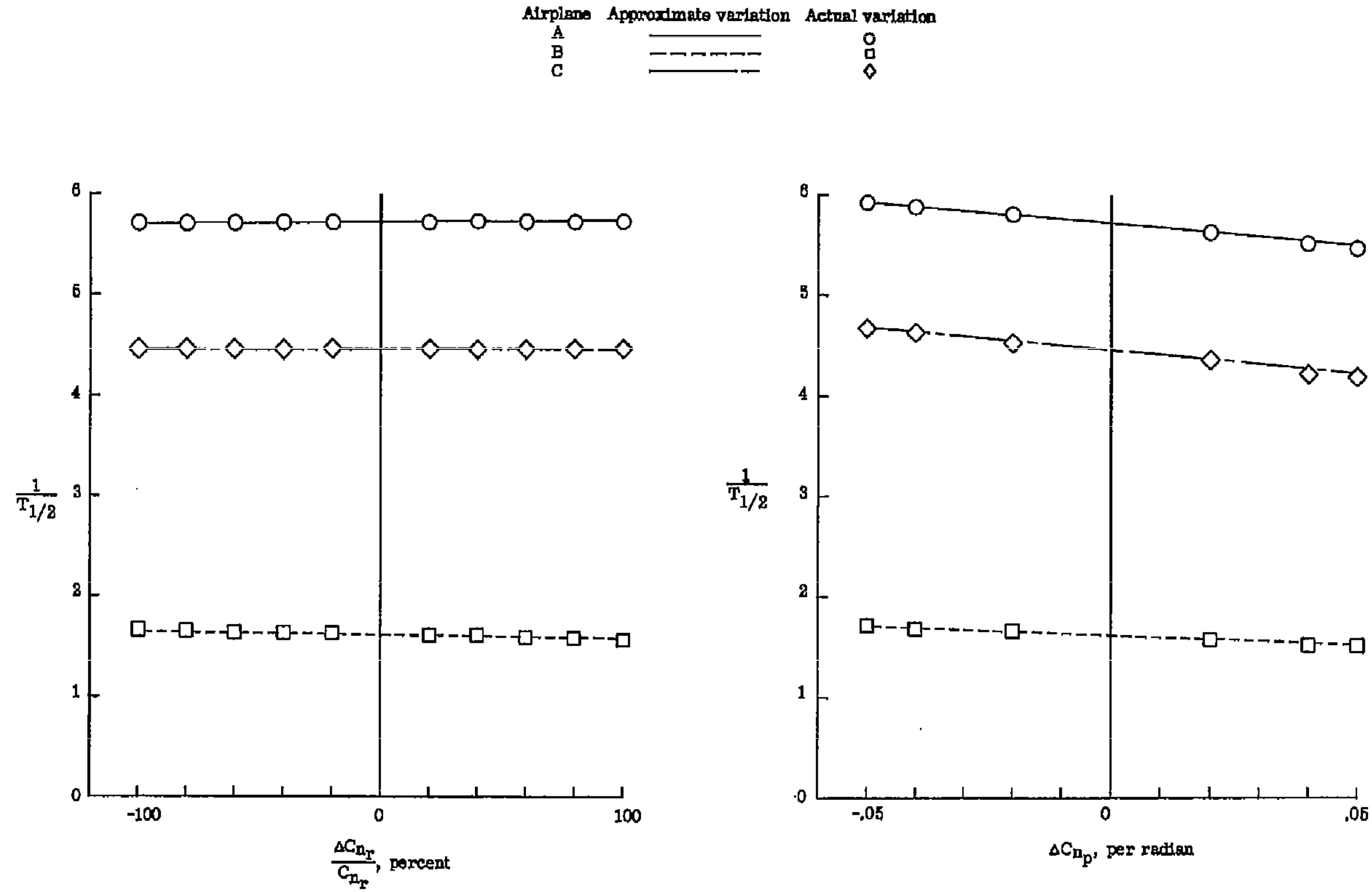
(e) $K_{Z_o}{}^2$ and $K_{X_o}{}^2$.

Figure 1.- Concluded.



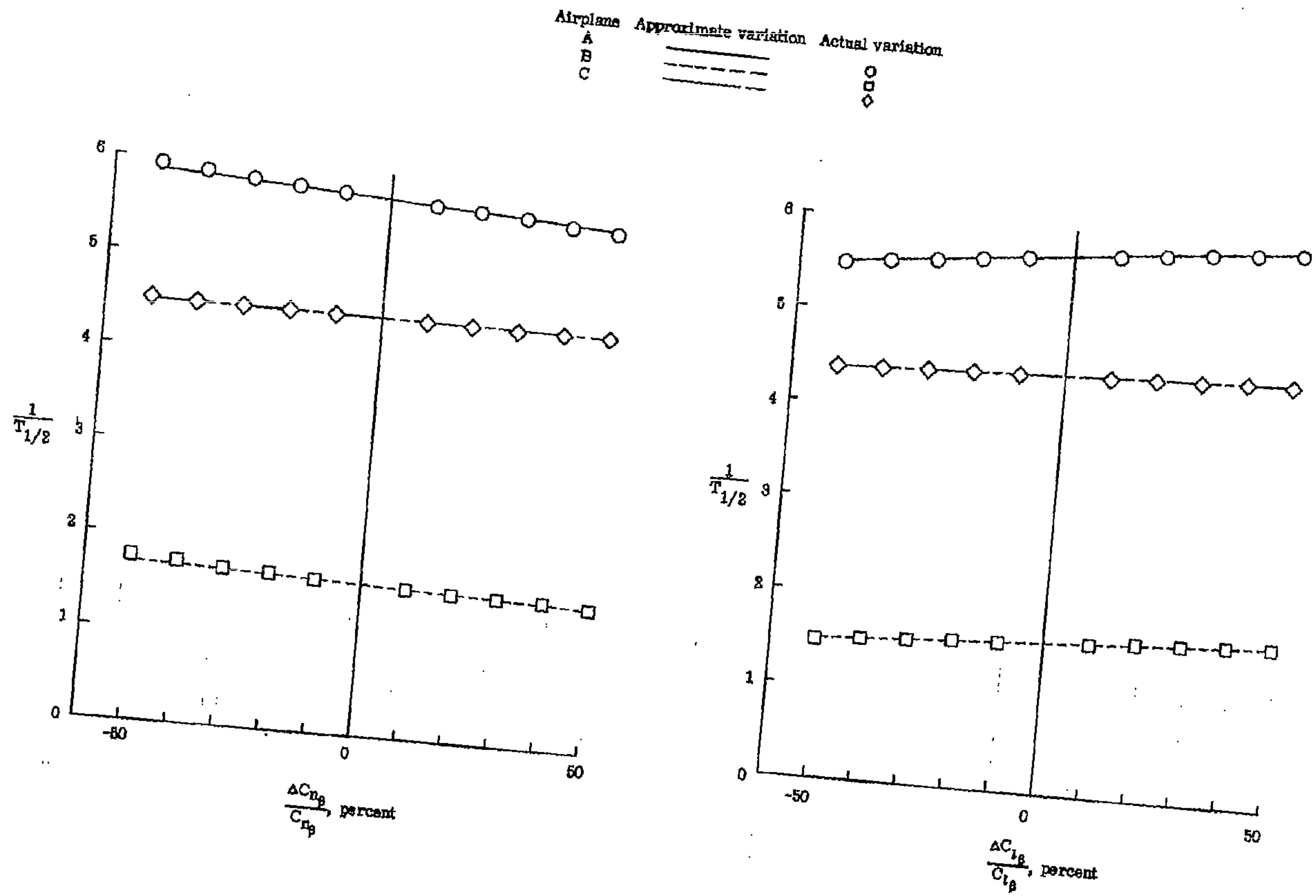
(a) C_{l_p} and C_{l_r}.

Figure 2.- Changes in the $1/T_{1/2}$ of the damping-in-roll root of three airplanes due to increments in various airplane parameters.



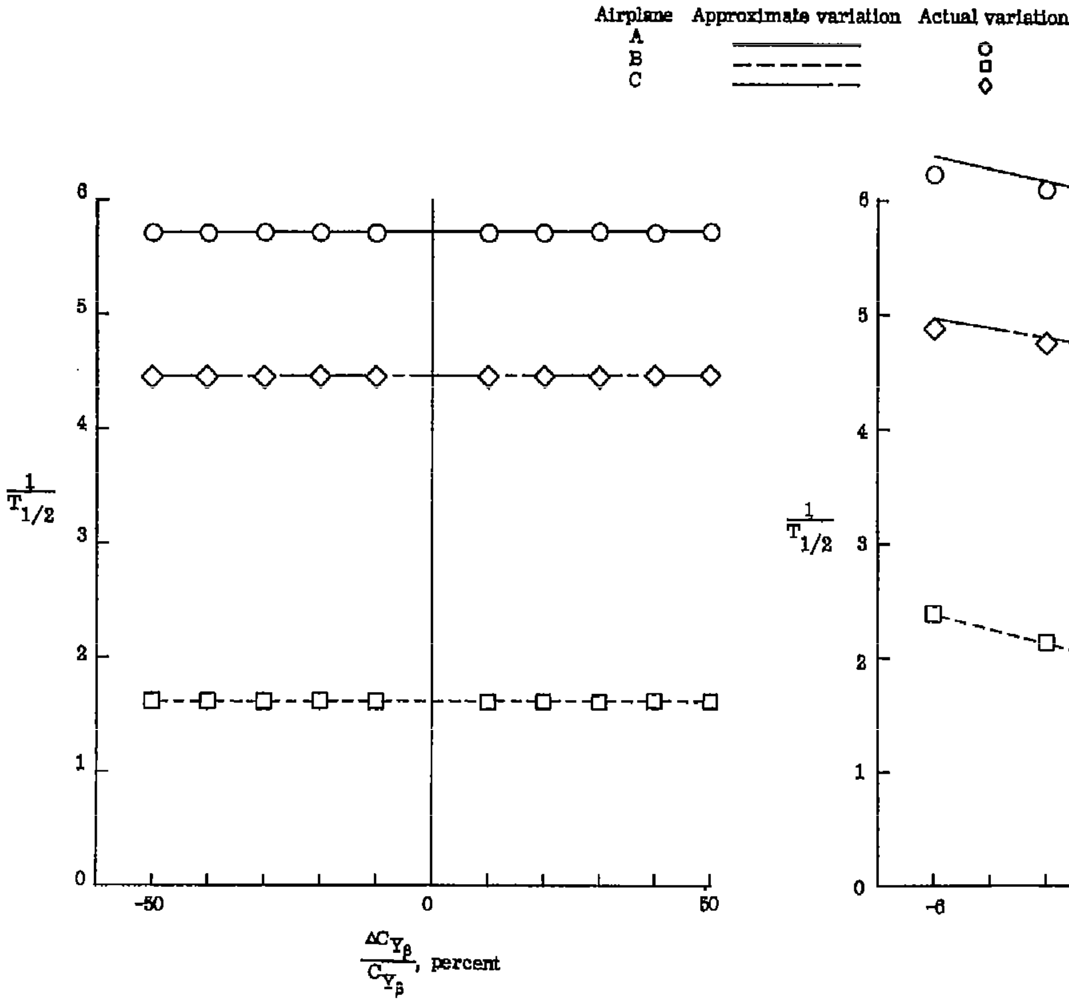
(b) C_{n_r} and C_{n_p}.

Figure 2.- Continued.



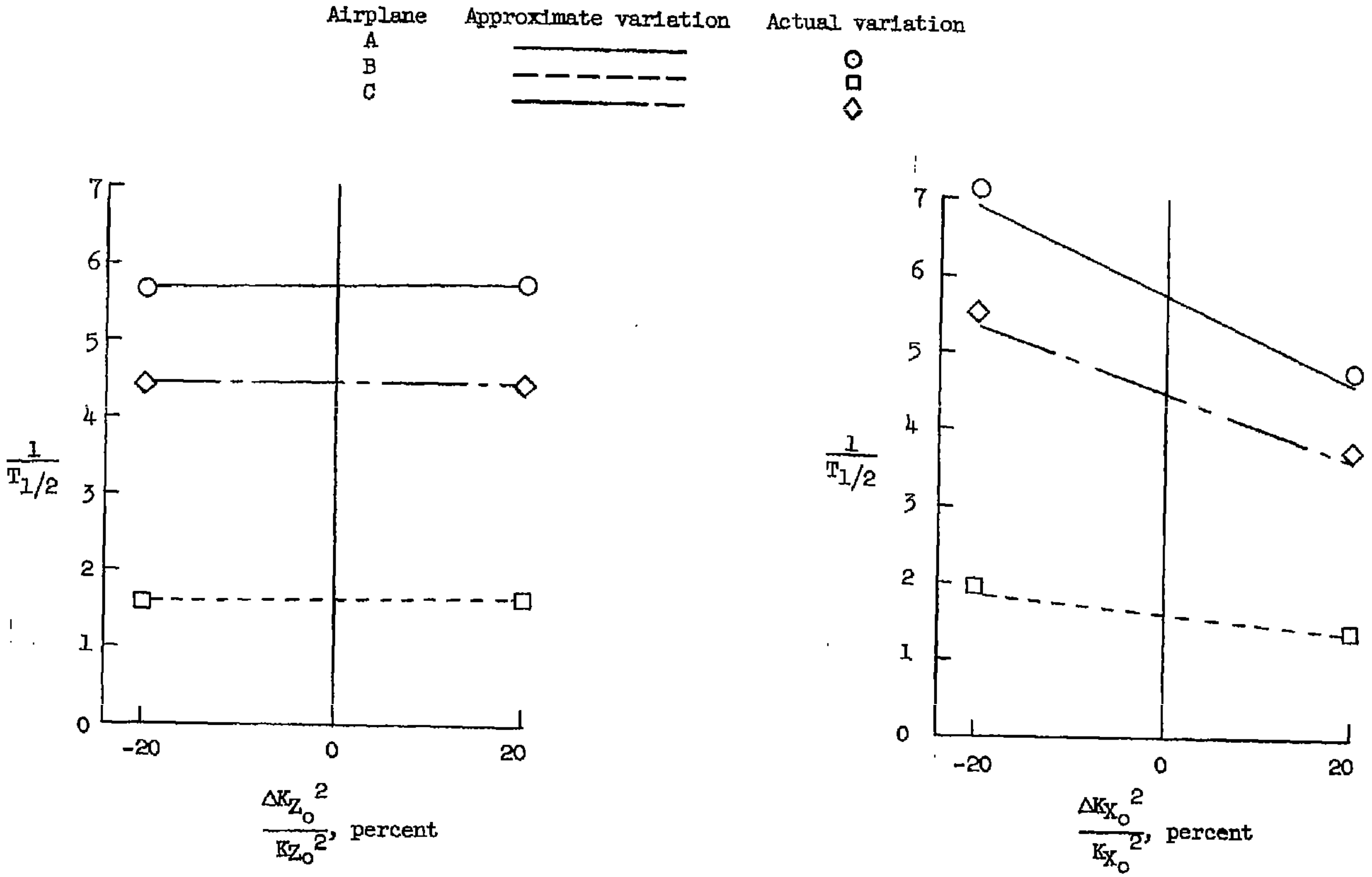
(c) C_{n_β} and C_{l_β}.

Figure 2.- Continued.



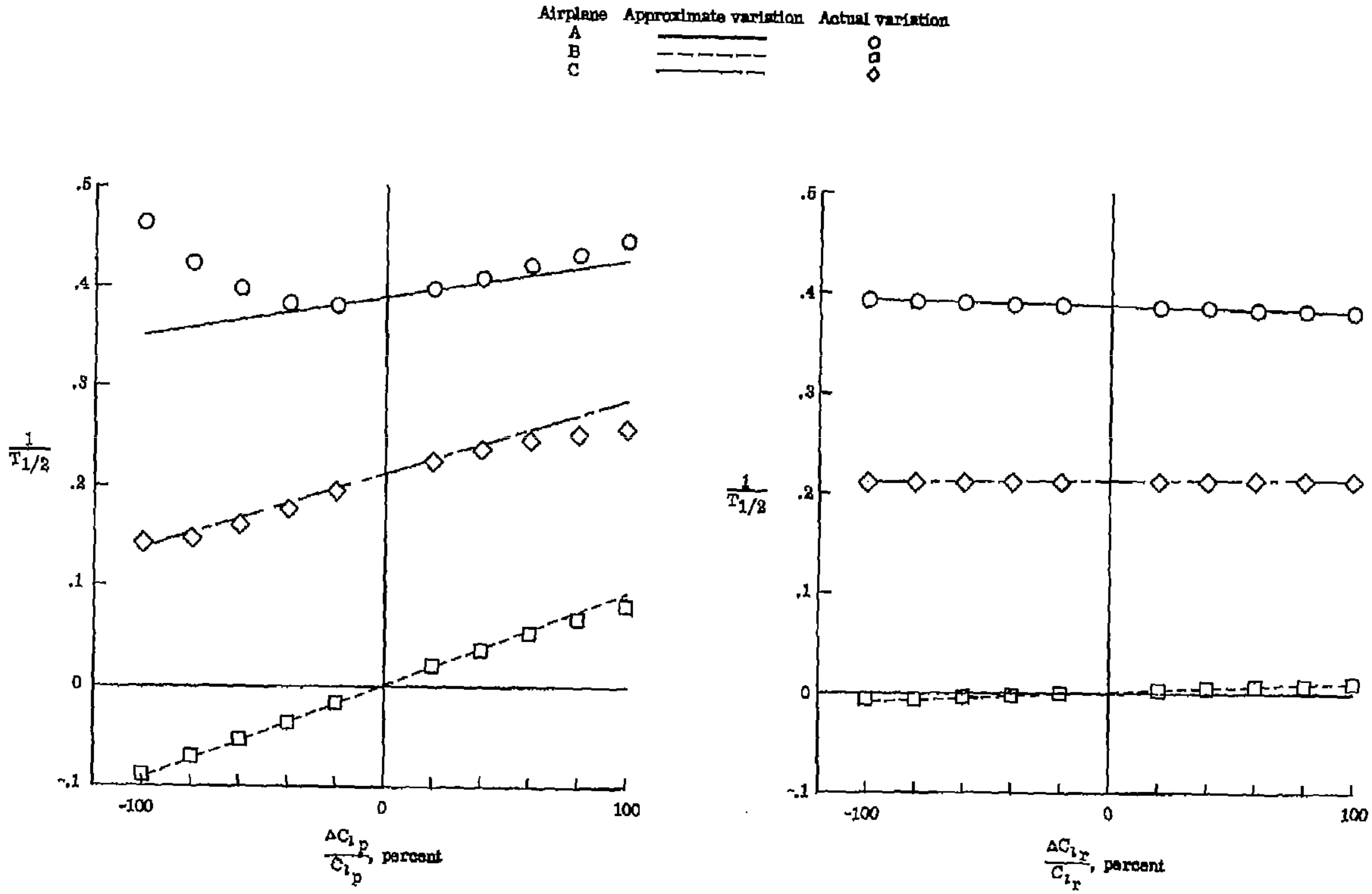
(d) C_{Y_β} and η.

Figure 2.- Continued.



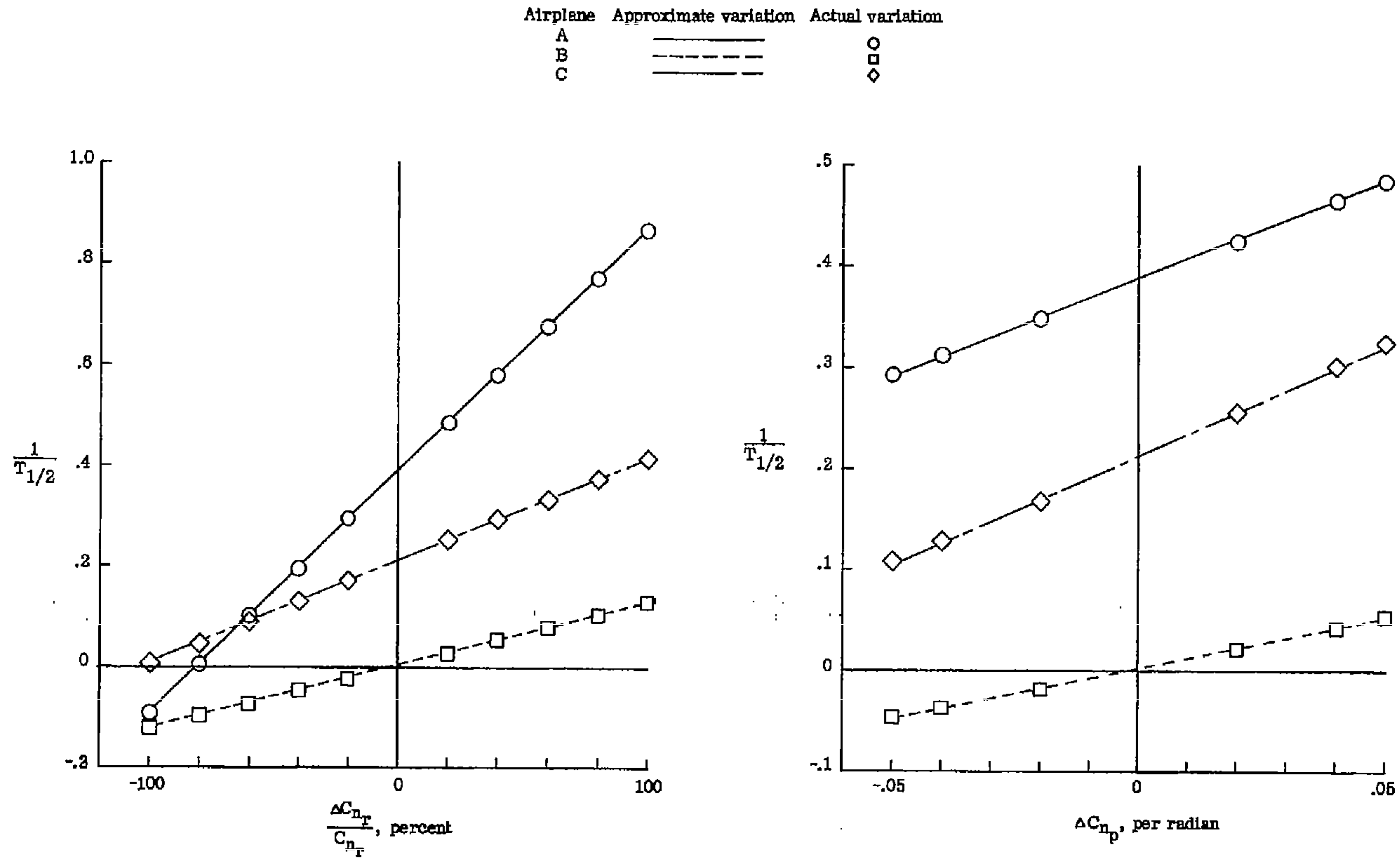
(e) $K_{Z_o}{}^2$ and $K_{X_o}{}^2$.

Figure 2.- Concluded.



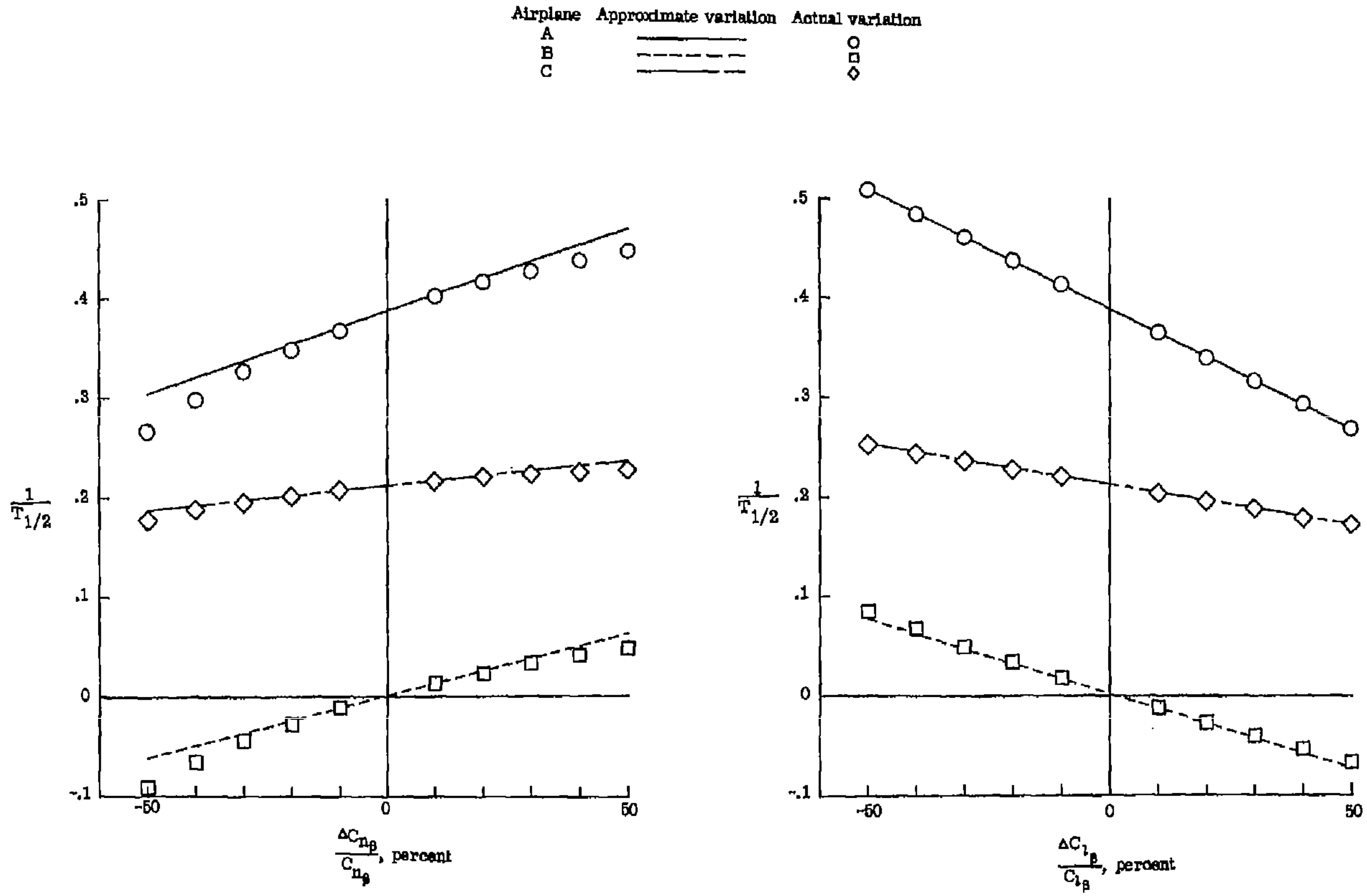
(a) C_{l_p} and C_{l_r}.

Figure 3.- Changes in the $1/T_{1/2}$ of the Dutch roll oscillation of three airplanes due to increments in various airplane parameters.



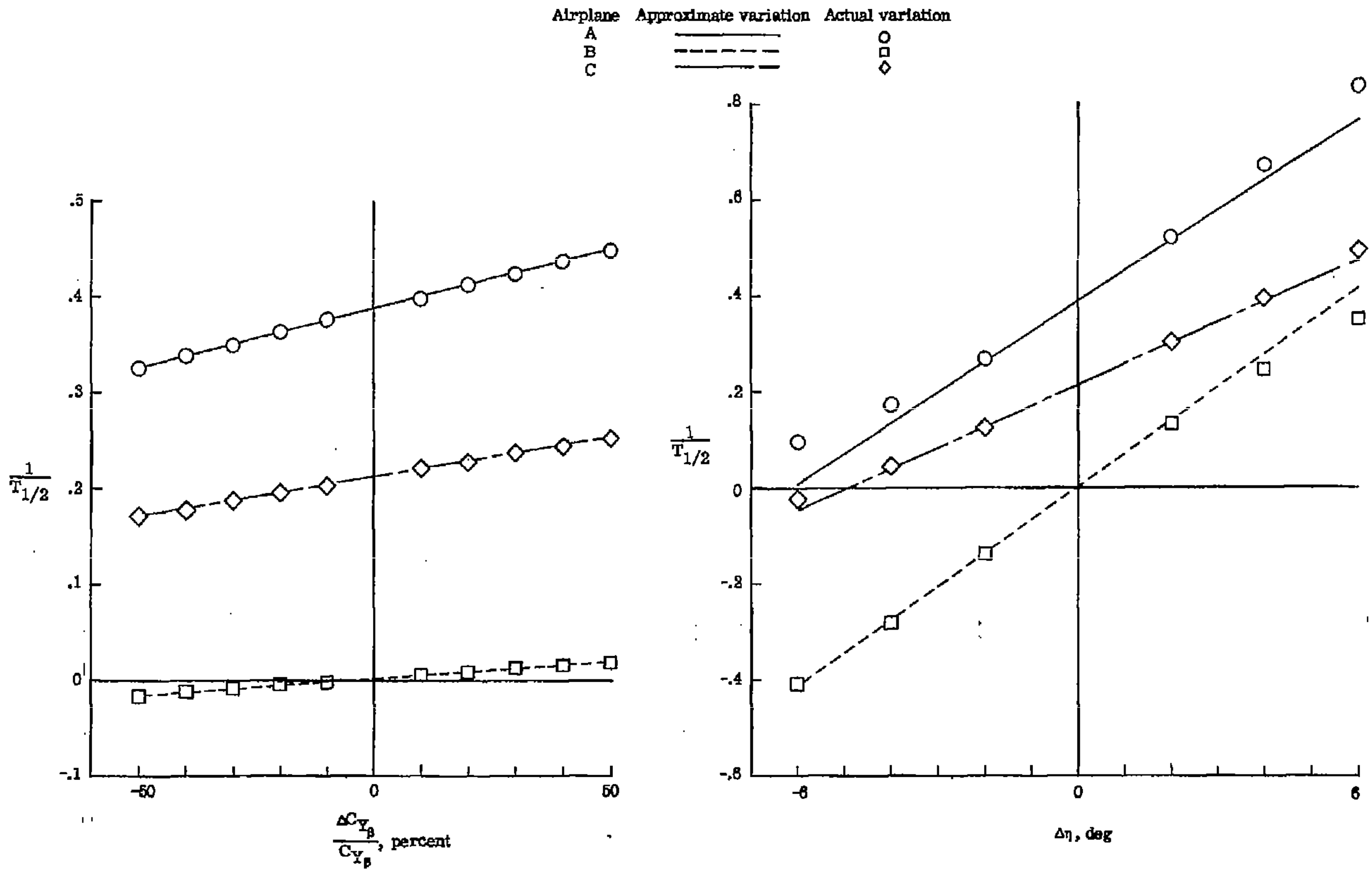
(b) C_{n_r} and C_{n_p}.

Figure 3.- Continued.



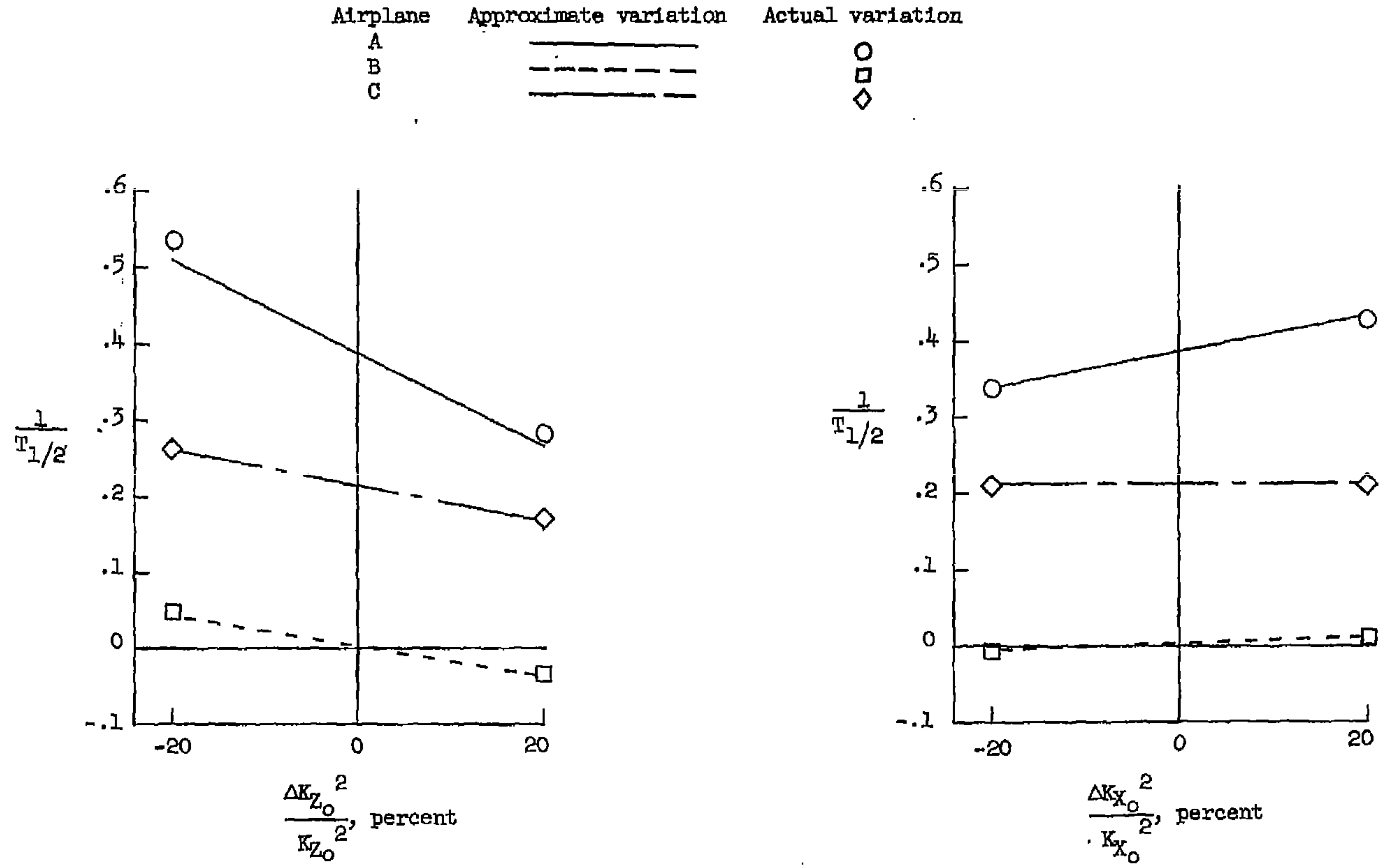
(c) C_{n_β} and C_{l_β}.

Figure 3.- Continued.



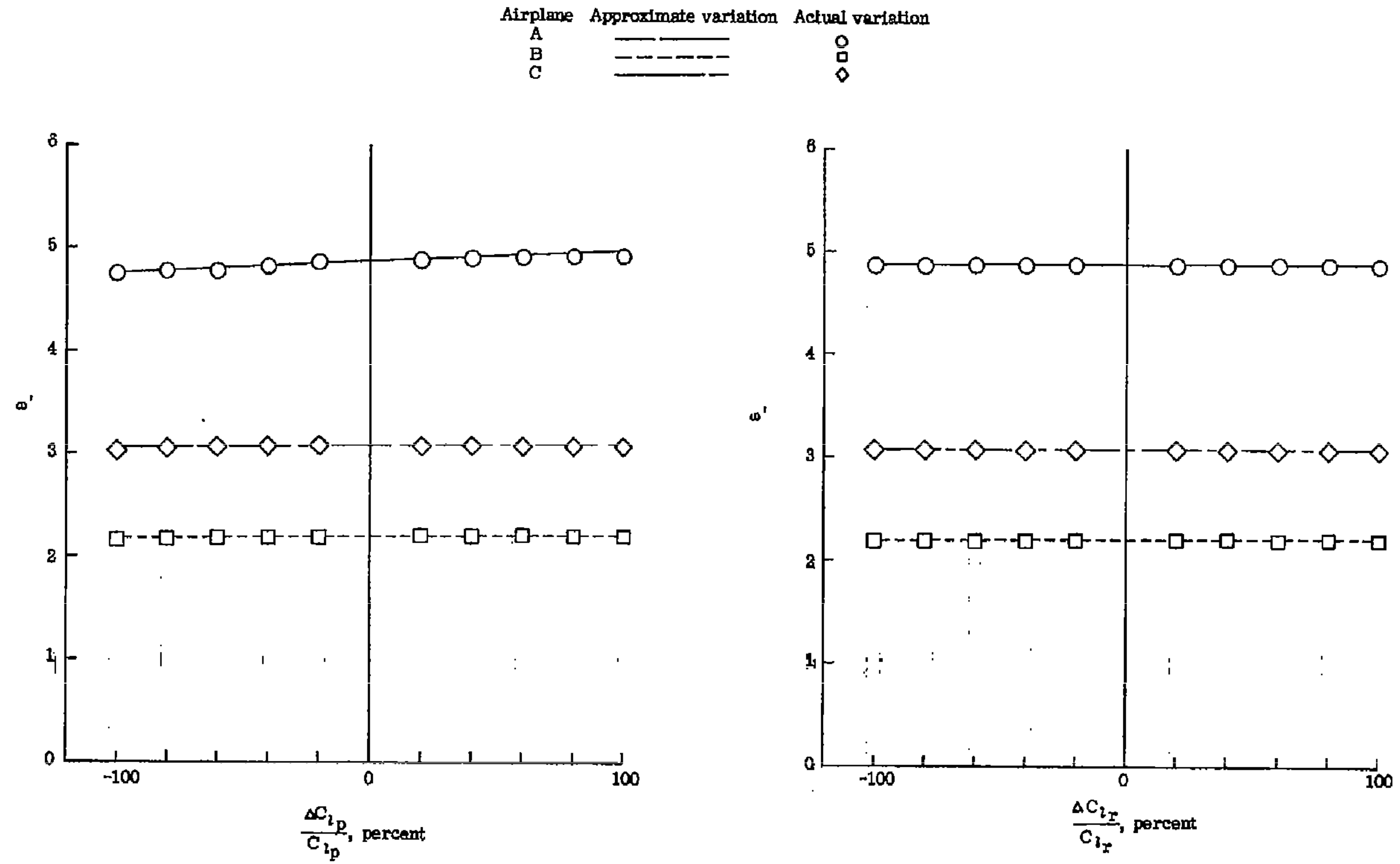
(d) C_{Y_β} and η.

Figure 3.- Continued.



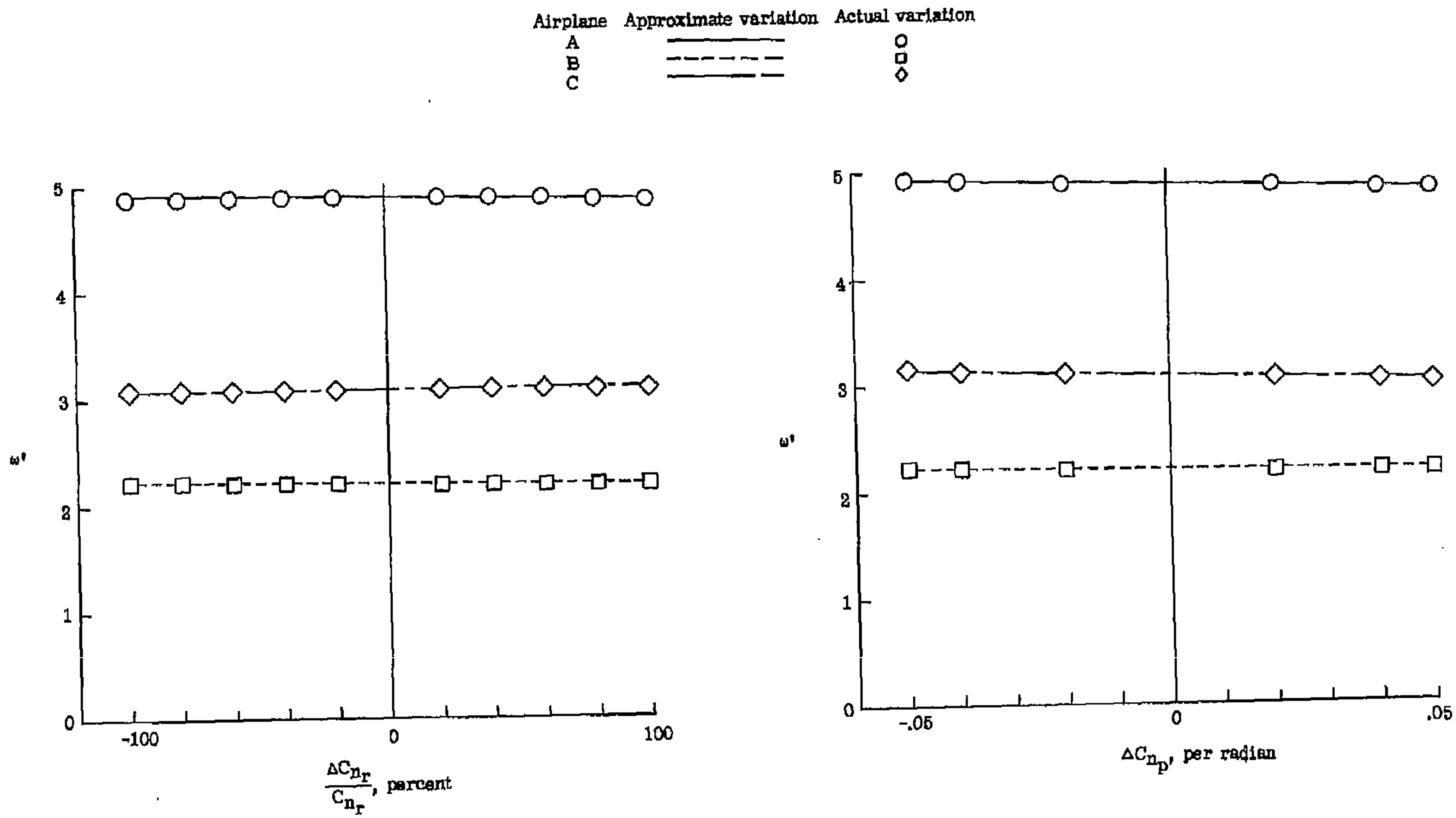
(e) $K_{Z_0}^{2}$ and $K_{X_0}^{2}$.

Figure 3.- Concluded.



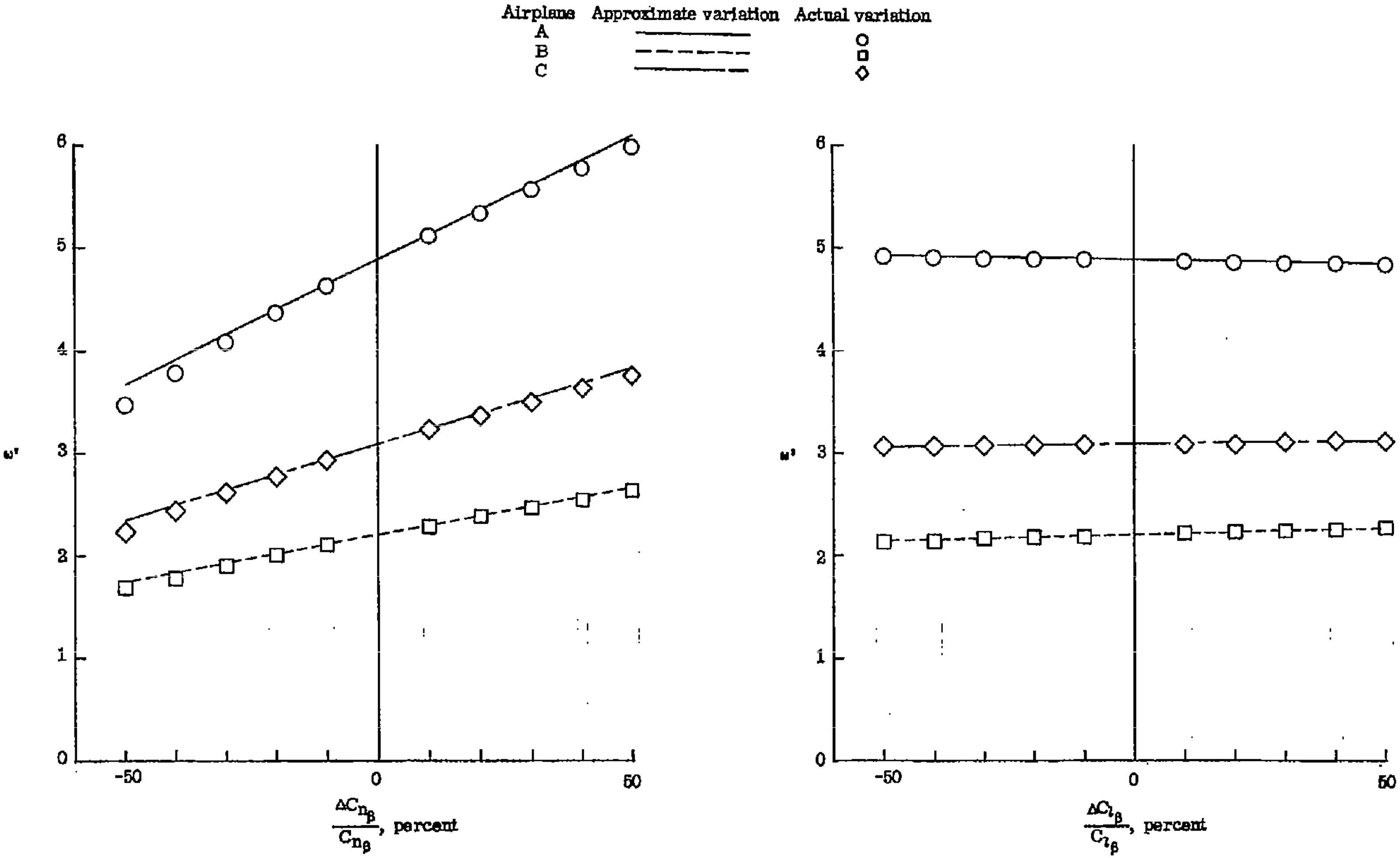
(a) C_{l_p} and C_{l_r}.

Figure 4.- Changes in the frequency of the Dutch roll oscillation of three airplanes due to increments in various airplane parameters.



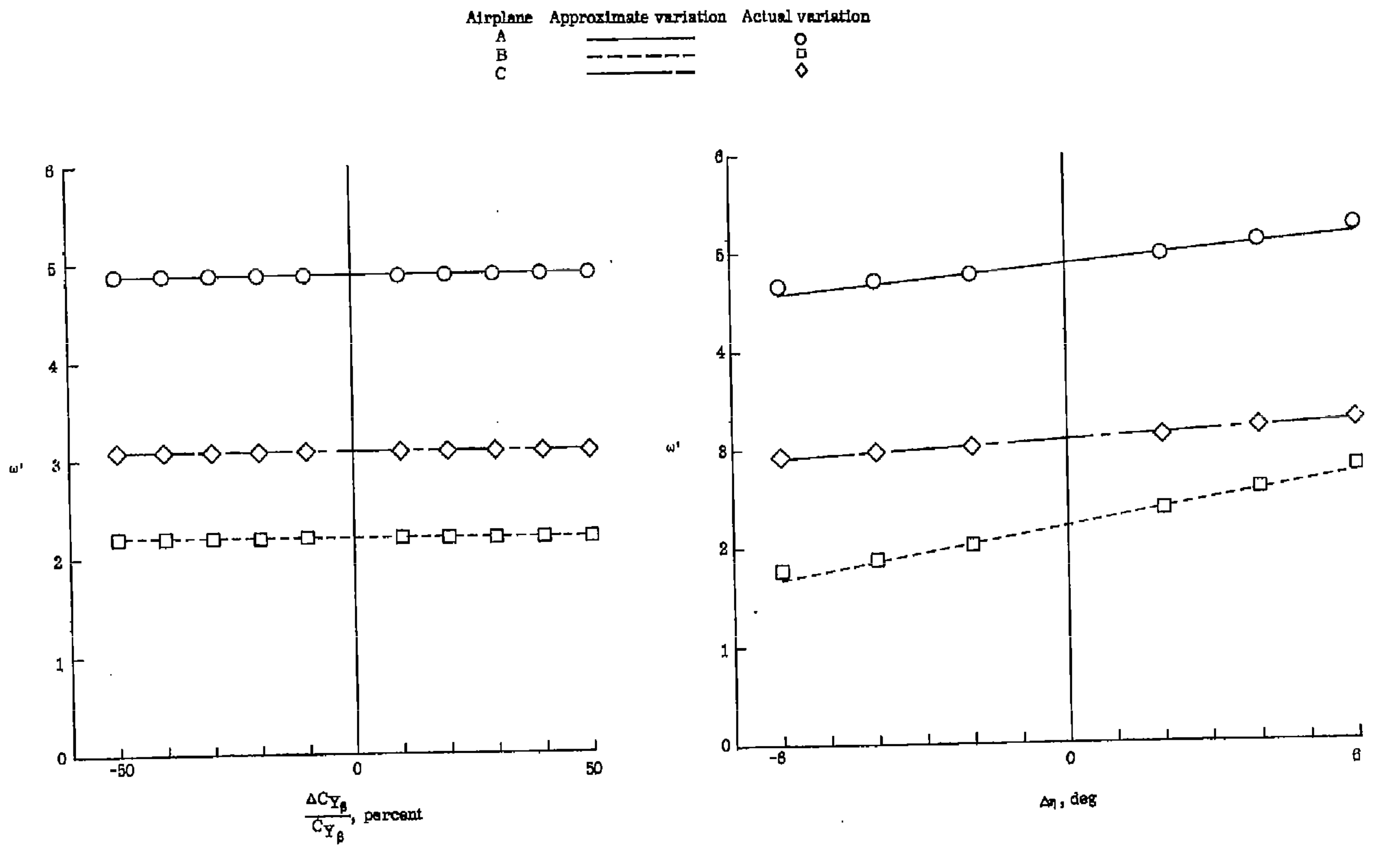
(b) C_{n_r} and C_{n_p}.

Figure 4.- Continued.



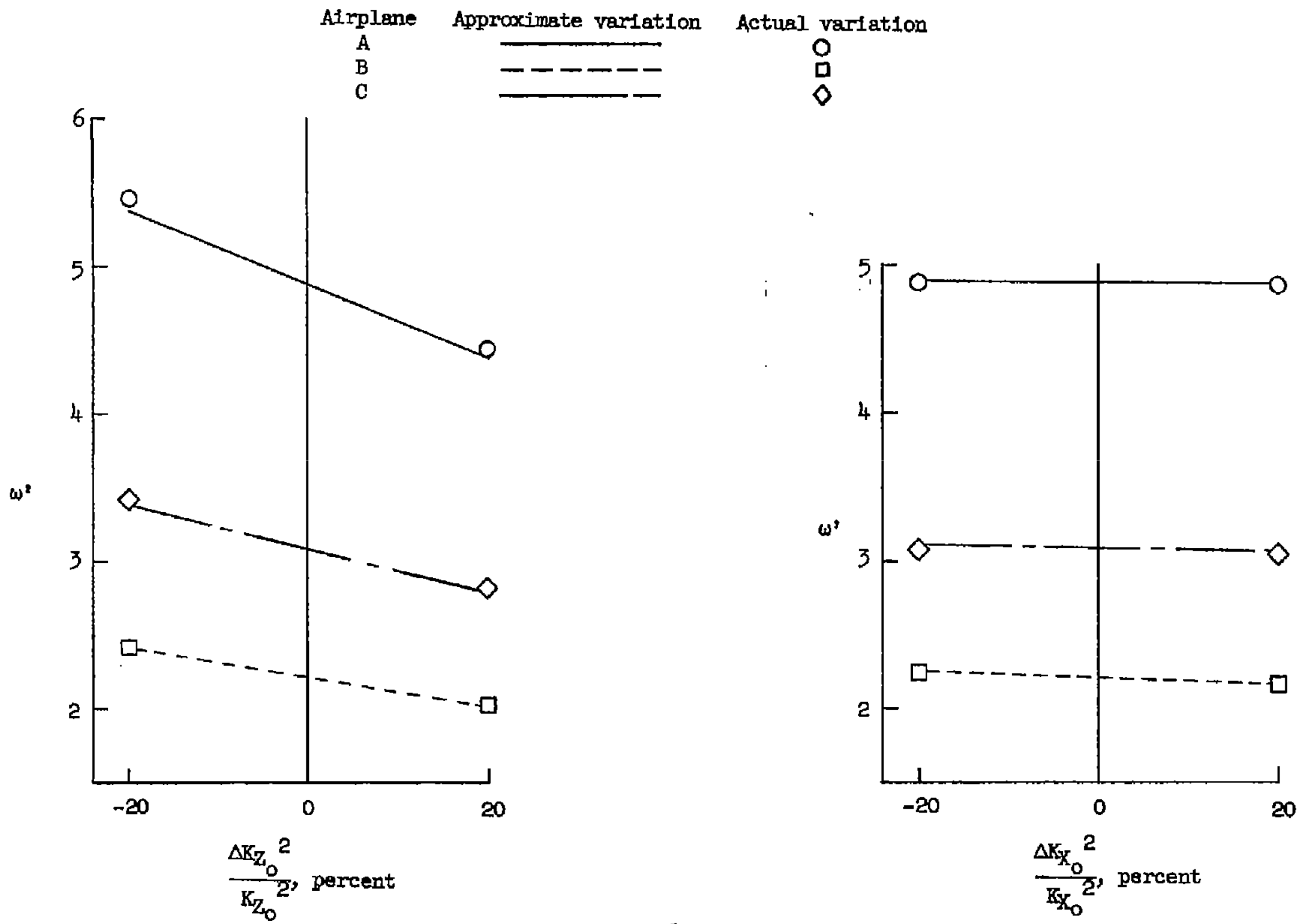
(c) C_{n_β} and C_{l_β}.

Figure 4.- Continued.



(d) C_{Y_β} and η.

Figure 4.- Continued.



(e) $K_{Z_o}^{2}$ and $K_{X_o}^{2}$.

Figure 4.- Concluded.

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